Emerging of massive gauge particles in inhomogeneous local gauge transformations: replacement of Higgs mechanism

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Abstract. A generalised theory of gauge transformations is presented on the basis of the covariant Hamiltonian formalism of field theory, for which the covariant canonical field equations are equivalent to the Euler-Lagrange field equations. Similar to the canonical transformation theory of point dynamics, the canonical transformation rules for fields are derived from generating functions. Thus—in contrast to the usual Lagrangian description—the covariant canonical transformation formalism automatically ensures the mappings to preserve the action principle, and hence to be *physical*. On that basis, we work out the theory of *inhomogeneous* local gauge transformations that generalises the conventional local $\mathrm{SU}(N)$ gauge transformation theory. It is shown that massive gauge bosons naturally emerge in this description, which thus could supersede the Higgs mechanism.

"Die Fruchtbarkeit des neuen Gesichtspunktes der Eichinvarianz hätte sich vor allem am Problem der Materie zu zeigen." (Weyl 1919)
"The fruitfulness of the new viewpoint of gauge invariance would have to show up especially on the problem of matter."

1. Introduction

The principle of *local gauge invariance* has been proven to be an eminently fruitful device for deducing all elementary particle interactions within the standard model. On the other hand, the gauge principle is justified only as far as it "works": a deeper rationale underlying the gauge principle apparently does not exist. In this respect, the gauge principle corresponds to other basic principles of physics, such as Fermat's "principle of least time," the "principle of least action" as well as its quantum generalisation leading to Feynman's path integral formalism. The failure of the conventional gauge principle to explain the existence of massive gauge bosons has led to *supplementing* it with the Higgs-Kibble mechanism (Higgs 1964, Kibble 1967).

An alternative strategy to resolve the mass problem would be to directly generalise the conventional gauge principle in a natural way. One way to achieve this was to require the system's covariant Hamiltonian to be form-invariant not only under unitary transformations of the fields in iso-space, but also under variations of the space-time metric. This idea of a generalisation of the conventional gauge principle has been successfully worked out and

was published recently (Struckmeier 2013). In this description, the gauge field causes a non-vanishing curvature tensor, and this curvature tensor appears in the field equations as a mass factor.

With the actual paper, a second natural generalisation of the conventional gauge transformation formalism will be presented that extends the conventional $\mathrm{SU}(N)$ gauge theory to include *inhomogeneous* linear mappings of the fields. As it turns out, the local gauge-invariance of the system's Lagrangian then requires the existence of massive gauge fields, with the mass playing the role of a second coupling constant. We thereby tackle long-standing inconsistency of the *conventional* gauge principle that requires gauge bosons to be massless in order for any theory to be locally gauge-invariant. This will be achieved *without* postulating a particular potential function ("Mexican hat") and without requiring a "symmetry breaking" phenomenon.

Conventional gauge theories are commonly derived on the basis of Lagrangians of relativistic field theory (cf, for instance, Ryder 1996, Griffiths 2008, Cheng and Li 2000). Although perfectly valid, the Lagrangian formulation of gauge transformation theory is *not* the optimum choice. The reason is that in order for a Lagrangian transformation theory to be physical, hence to maintain the action principle, it must be supplemented by additional structure, referred to as the *minimum coupling rule*.

In contrast, the formulation of gauge theories in terms of *covariant Hamiltonians*—each of them being equivalent to a corresponding Lagrangian—may exploit the framework of the *canonical transformation formalism*. With the transformation rules for all fields and their canonical conjugates being derived from *generating functions*, we restrict ourselves from the outset to exactly the *subset* of transformations that preserve the action principle, hence ensure the actual gauge transformation to be *physical*. No additional structure needs to be incorporated for setting up an amended Hamiltonian that is *locally* gauge-invariant on the basis of a given *globally* gauge-invariant Hamiltonian. Furthermore, it is no longer required to postulate the field tensor to be skew-symmetric in its space-time indices as this feature directly emerges from the canonical transformation formalism.

Prior to working out the inhomogeneous local gauge transformation theory in the covariant Hamiltonian formalism—the latter dating back to DeDonder (DeDonder 1930) and Weyl (Weyl 1935)—a concise review of the concept of covariant Hamiltonians in local coordinate representation is outlined in section 2. Thereafter, the canonical transformation formalism in the realm of field theory is sketched briefly in section 3. In these sections, we restrict our presentation to exactly those topics of the canonical formalism that are essential for working out the inhomogeneous gauge transformation theory, which will finally be covered in section 4.

The requirement of *inhomogeneous* local gauge invariance naturally generalises the conventional $\mathrm{SU}(N)$ gauge principle (cf, for instance, Struckmeier and Reichau 2012), where the form-invariance of the covariant Hamiltonian density is demanded under *homogeneous* unitary mappings of the fields in iso-space. In the first step, a generating function of type $\boldsymbol{F}_2$ is set up that merely describes the demanded transformation of the fields in iso-space. As usual, this transformation forces us to introduce gauge fields that render an appropriately amended

Hamiltonian locally gauge-invariant if the gauge fields follow a particular transformation law. In our case of an inhomogeneous mapping, we are forced to introduce *two* independent sets of gauge fields, each of them requiring its own transformation law.

In the second step, an *amended* generating function $\boldsymbol{F}_2$ is constructed in a way to define these transformation laws for the two sets of gauge fields in addition to the rules for the base fields. As the characteristic feature of the canonical transformation formalism, this amended generating function also provides the transformation law for the conjugates of the gauge fields and for the Hamiltonian. This way, we derive the Hamiltonian that is form-invariant under both the inhomogeneous mappings of the base fields as well as under the required mappings of the two sets of gauge fields.

In a third step, it must be ensured that the canonical field equations emerging from the gauge-invariant Hamiltonian are consistent with the canonical transformation rules. As usual in gauge theories, the Hamiltonian must be further amended by terms that describe the free-field dynamics of the gauge fields while maintaining the overall form-invariance of the final Hamiltonian. Amazingly, this also works for our inhomogeneous gauge transformation theory and *uniquely* determines the final gauge-invariant Hamiltonian $\mathcal{H}_3$.

The Hamiltonian $\mathcal{H}_3$ is then Legendre-transformed to yield the equivalent gauge-invariant Lagrangian density $\mathcal{L}_3$. The latter can then serve as the starting point to set up the Feynman diagrams for the various mutual interactions of base and gauge fields. As an example, the gauge-invariant Lagrangian that emerges from a base system of an N-tuple of massless spin-0 fields is presented.

2. Covariant Hamiltonian density

In field theory, the Hamiltonian density is usually defined by performing an *incomplete* Legendre transformation of a Lagrangian density $\mathcal{L}$ that only maps the time derivative $\partial_t\phi$ of a field $\phi(t,x,y,z)$ into a corresponding canonical momentum variable, π_t. Taking then the spatial integrals results in a description that corresponds to that of non-relativistic Hamiltonian point dynamics. Yet, in analogy to relativistic point dynamics (Struckmeier 2009), a covariant Hamiltonian description of field theory must treat space and time variables on equal footing. If $\mathcal{L}$ is a Lorentz scalar, this property is passed to the *covariant DeDonder-Weyl Hamiltonian density* $\mathcal{H}$ that emerges from a *complete* Legendre transformation of $\mathcal{L}$. Moreover, this description enables us to devise a consistent theory of canonical transformations in the realm of classical field theory.

2.1. Covariant canonical field equations

The transition from particle dynamics to the dynamics of a *continuous* system is based on the assumption that a *continuum limit* exists for the given physical problem (José and Saletan 1998). This limit is defined by letting the number of particles involved in the system increase over all bounds while letting their masses and distances go to zero. In this limit, the information on the location of individual particles is replaced by the *value* of a smooth

function $\phi(x)$ that is given at a spatial location x^1, x^2, x^3 at time $t \equiv x^0$. In this notation, the index μ runs from 0 to 3, hence distinguishes the four independent variables of space-time $x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$, and $x_\mu \equiv (x_0, x_1, x_2, x_3) \equiv (t, -x, -y, -z)$. We furthermore assume that the given physical problem can be described in terms of a set of $I = 1, \ldots, N$—possibly interacting—scalar fields $\phi_I(x)$, with the index "I" enumerating the individual fields. A transformation of the fields in iso-space is not associated with any non-trivial metric. We, therefore, do not use superscripts for these indices as there is not distinction between covariant and contravariant components. In contrast, Greek indices are used for those components that *are* associated with a metric—such as the derivatives with respect to a space-time variable, x^μ. Throughout the article, the summation convention is used. Whenever no confusion can arise, we omit the indices in the argument list of functions in order to avoid the number of indices to proliferate.

The Lagrangian description of the dynamics of a continuous system is based on the Lagrangian density function $\mathcal{L}$ that is supposed to carry the complete information on the given physical system. In a first-order field theory, the Lagrangian density $\mathcal{L}$ is defined to depend on the ϕ_I, possibly on the vector of independent variables x^μ, and on the four first derivatives of the fields ϕ_I with respect to the independent variables, i.e., on the 1-forms (covectors)

$$\partial_\mu \phi_I \equiv (\partial_t \phi_I, \partial_x \phi_I, \partial_y \phi_I, \partial_z \phi_I).$$

The Euler-Lagrange field equations are then obtained as the zero of the variation δS of the action integral

$$S = \int \mathcal{L}(\phi_I, \partial_\mu \phi_I, x)\, d^4x \tag{1}$$

as

$$\frac{\partial}{\partial x^\alpha} \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_I)} - \frac{\partial \mathcal{L}}{\partial \phi_I} = 0. \tag{2}$$

To derive the equivalent *covariant* Hamiltonian description of continuum dynamics, we first define for each field $\phi_I(x)$ a 4-vector of conjugate momentum fields $\pi_I^\mu(x)$. Its components are given by

$$\pi_I^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_I)} \equiv \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi_I}{\partial x^\mu}\right)}. \tag{3}$$

The 4-vector $\boldsymbol{\pi}_I$ is thus induced by the Lagrangian $\mathcal{L}$ as the *dual counterpart* of the 1-form $\partial_\mu \phi_I$. For the entire set of N scalar fields $\phi_I(x)$, this establishes a set of N conjugate 4-vector fields. With this definition of the 4-vectors of canonical momenta $\boldsymbol{\pi}_I(x)$, we can now define the Hamiltonian density $\mathcal{H}(\phi_I, \boldsymbol{\pi}_I, x)$ as the covariant Legendre transform of the Lagrangian density $\mathcal{L}(\phi_I, \partial_\mu \phi_I, x)$

$$\mathcal{H}(\phi_I, \boldsymbol{\pi}_I, x) = \pi_J^\alpha \frac{\partial \phi_J}{\partial x^\alpha} - \mathcal{L}(\phi_I, \partial_\mu \phi_I, x). \tag{4}$$

In order for the Hamiltonian $\mathcal{H}$ to be valid, we must require the Legendre transformation to be *regular*, which means that for each index "I" the Hesse matrices $(\partial^2 \mathcal{L} / \partial(\partial^\mu \phi_I)\, \partial(\partial_\nu \phi_I))$ are non-singular. This ensures that by means of the Legendre transformation, the Hamiltonian $\mathcal{H}$

takes over the complete information on the given dynamical system from the Lagrangian $\mathcal{L}$. The definition of $\mathcal{H}$ by Eq. (4) is referred to in literature as the "DeDonder-Weyl" Hamiltonian density.

Obviously, the dependencies of $\mathcal{H}$ and $\mathcal{L}$ on the ϕ_I and the x^μ only differ by a sign,

$$\left.\frac{\partial \mathcal{H}}{\partial x^\mu}\right|_{\text{expl}} = -\left.\frac{\partial \mathcal{L}}{\partial x^\mu}\right|_{\text{expl}}, \qquad \frac{\partial \mathcal{H}}{\partial \phi_I} = -\frac{\partial \mathcal{L}}{\partial \phi_I} = -\frac{\partial}{\partial x^\alpha}\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_I)} = -\frac{\partial \pi_I^\alpha}{\partial x^\alpha}.$$

These variables thus do not take part in the Legendre transformation of Eqs. (3), (4). Thus, with respect to this transformation, the Lagrangian density $\mathcal{L}$ represents a function of the $\partial_\mu \phi_I$ only and does *not depend* on the canonical momenta π_I^μ, whereas the Hamiltonian density $\mathcal{H}$ is to be considered as a function of the π_I^μ only and does not depend on the derivatives $\partial_\mu \phi_I$ of the fields. In order to derive the second canonical field equation, we calculate from Eq. (4) the partial derivative of $\mathcal{H}$ with respect to π_I^μ,

$$\frac{\partial \mathcal{H}}{\partial \pi_I^\mu} = \delta_{IJ}\,\delta_\mu^\alpha\,\frac{\partial \phi_J}{\partial x^\alpha} = \frac{\partial \phi_I}{\partial x^\mu} \qquad \Longleftrightarrow \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_I)} = \pi_J^\alpha \delta_{IJ}\,\delta_\alpha^\mu = \pi_I^\mu.$$

The complete set of covariant canonical field equations is thus given by

$$\frac{\partial \mathcal{H}}{\partial \pi_I^\mu} = \frac{\partial \phi_I}{\partial x^\mu}, \qquad \frac{\partial \mathcal{H}}{\partial \phi_I} = -\frac{\partial \pi_I^\alpha}{\partial x^\alpha}. \tag{5}$$

This pair of first-order partial differential equations is equivalent to the set of second-order differential equations of Eq. (2). We observe that in this formulation of the canonical field equations, all coordinates of space-time appear symmetrically—similar to the Lagrangian formulation of Eq. (2). Provided that the Lagrangian density $\mathcal{L}$ is a Lorentz scalar, the dynamics of the fields is invariant with respect to Lorentz transformations. The covariant Legendre transformation (4) passes this property to the Hamiltonian density $\mathcal{H}$. It thus ensures *a priori* the relativistic invariance of the fields that emerge as integrals of the canonical field equations if $\mathcal{L}$—and hence $\mathcal{H}$—represents a Lorentz scalar.

3. Canonical transformations in covariant Hamiltonian field theory

The covariant Legendre transformation (4) allows us to derive a canonical transformation theory in a way similar to that of point dynamics. The main difference is that now the generating function of the canonical transformation is represented by a *vector* rather than by a scalar function. The main benefit of this formalism is that we are not dealing with plain transformations. Instead, we restrict ourselves *right from the beginning* to those transformations that preserve the form of the action functional. This ensures all eligible transformations to be *physical*. Furthermore, with a generating function, we not only define the transformations of the fields but also pinpoint simultaneously the corresponding transformation law of the canonical momentum fields.

3.1. Generating functions of type $\boldsymbol{F}_1(\boldsymbol{\phi}, \boldsymbol{\Phi}, x)$

Similar to the canonical formalism of point mechanics, we call a transformation of the fields $(\boldsymbol{\phi}, \boldsymbol{\pi}) \mapsto (\boldsymbol{\Phi}, \boldsymbol{\Pi})$ *canonical* if the form of the variational principle that is based on the action functional (1) is maintained,

$$\delta \int_R \left(\pi_I^\alpha \frac{\partial \phi_I}{\partial x^\alpha} - \mathcal{H}(\boldsymbol{\phi}, \boldsymbol{\pi}, x) \right) d^4x \stackrel{!}{=} \delta \int_R \left(\Pi_I^\alpha \frac{\partial \Phi_I}{\partial x^\alpha} - \mathcal{H}'(\boldsymbol{\Phi}, \boldsymbol{\Pi}, x) \right) d^4x. \quad (6)$$

Equation (6) tells us that the *integrands* may differ by the divergence of a vector field F_1^μ, whose variation vanishes on the boundary ∂R of the integration region R within space-time

$$\delta \int_R \frac{\partial F_1^\alpha}{\partial x^\alpha} d^4x = \delta \oint_{\partial R} F_1^\alpha dS_\alpha \stackrel{!}{=} 0.$$

The immediate consequence of the form invariance of the variational principle is the form invariance of the covariant canonical field equations (5)

$$\frac{\partial \mathcal{H}'}{\partial \Pi_I^\mu} = \frac{\partial \Phi_I}{\partial x^\mu}, \qquad \frac{\partial \mathcal{H}'}{\partial \Phi_I} = -\frac{\partial \Pi_I^\alpha}{\partial x^\alpha}.$$

For the integrands of Eq. (6)—hence for the Lagrangian densities $\mathcal{L}$ and $\mathcal{L}'$—we thus obtain the condition

$$\begin{aligned} &\mathcal{L} = \mathcal{L}' + \frac{\partial F_1^\alpha}{\partial x^\alpha} \\ &\pi_I^\alpha \frac{\partial \phi_I}{\partial x^\alpha} - \mathcal{H}(\boldsymbol{\phi}, \boldsymbol{\pi}, x) = \Pi_I^\alpha \frac{\partial \Phi_I}{\partial x^\alpha} - \mathcal{H}'(\boldsymbol{\Phi}, \boldsymbol{\Pi}, x) + \frac{\partial F_1^\alpha}{\partial x^\alpha}. \end{aligned} \quad (7)$$

With the definition $F_1^\mu \equiv F_1^\mu(\boldsymbol{\phi}, \boldsymbol{\Phi}, x)$, we restrict ourselves to a function of exactly those arguments that now enter into transformation rules for the transition from the original to the new fields. The divergence of F_1^μ writes, explicitly,

$$\frac{\partial F_1^\alpha}{\partial x^\alpha} = \frac{\partial F_1^\alpha}{\partial \phi_I} \frac{\partial \phi_I}{\partial x^\alpha} + \frac{\partial F_1^\alpha}{\partial \Phi_I} \frac{\partial \Phi_I}{\partial x^\alpha} + \left. \frac{\partial F_1^\alpha}{\partial x^\alpha} \right|_{\text{expl}}. \quad (8)$$

The rightmost term denotes the sum over the *explicit* dependence of the generating function F_1^μ on the x^ν. Comparing the coefficients of Eqs. (7) and (8), we find the local coordinate representation of the field transformation rules that are induced by the generating function F_1^μ

$$\pi_I^\mu = \frac{\partial F_1^\mu}{\partial \phi_I}, \qquad \Pi_I^\mu = -\frac{\partial F_1^\mu}{\partial \Phi_I}, \qquad \mathcal{H}' = \mathcal{H} + \left. \frac{\partial F_1^\alpha}{\partial x^\alpha} \right|_{\text{expl}}. \quad (9)$$

The transformation rule for the Hamiltonian density implies that summation over α is to be performed. In contrast to the transformation rule for the Lagrangian density $\mathcal{L}$ of Eq. (7), the rule for the Hamiltonian density is determined by the *explicit* dependence of the generating function F_1^μ on the x^ν. Hence, if a generating function does not explicitly depend on the independent variables, x^ν, then the *value* of the Hamiltonian density is not changed under the particular canonical transformation emerging thereof.

Differentiating the transformation rule for π_I^μ with respect to Φ_J, and the rule for Π_J^μ with respect to ϕ_I, we obtain a symmetry relation between original and transformed fields

$$\frac{\partial \pi_I^\mu}{\partial \Phi_J} = \frac{\partial^2 F_1^\mu}{\partial \phi_I \partial \Phi_J} = -\frac{\partial \Pi_J^\mu}{\partial \phi_I}.$$

The emerging of symmetry relations is a characteristic feature of *canonical* transformations. As the symmetry relation directly follows from the second derivatives of the generating function, is does not apply for arbitrary transformations of the fields that do not follow from generating functions.

3.2. Generating functions of type $\boldsymbol{F}_2(\boldsymbol{\phi}, \boldsymbol{\Pi}, x)$

The generating function of a canonical transformation can alternatively be expressed in terms of a function of the original fields ϕ_I and of the new *conjugate* fields Π_I^μ. To derive the pertaining transformation rules, we perform the covariant Legendre transformation

$$F_2^\mu(\boldsymbol{\phi}, \boldsymbol{\Pi}, x) = F_1^\mu(\boldsymbol{\phi}, \boldsymbol{\Phi}, x) + \Phi_J \Pi_J^\mu, \qquad \Pi_I^\mu = -\frac{\partial F_1^\mu}{\partial \Phi_I}. \tag{10}$$

By definition, the functions F_1^μ and F_2^μ agree with respect to their ϕ_I and x^μ dependencies

$$\frac{\partial F_2^\mu}{\partial \phi_I} = \frac{\partial F_1^\mu}{\partial \phi_I} = \pi_I^\mu, \qquad \left.\frac{\partial F_2^\alpha}{\partial x^\alpha}\right|_{\text{expl}} = \left.\frac{\partial F_1^\alpha}{\partial x^\alpha}\right|_{\text{expl}} = \mathcal{H}' - \mathcal{H}.$$

The variables ϕ_I and x^μ thus do not take part in the Legendre transformation from Eq. (10). Therefore, the two F_2^μ-related transformation rules coincide with the respective rules derived previously from F_1^μ. As F_1^μ does not depend on the Π_I^μ whereas F_2^μ does not depend on the the Φ_I, the new transformation rule thus follows from the derivative of F_2^μ with respect to Π_J^ν as

$$\frac{\partial F_2^\mu}{\partial \Pi_I^\nu} = \Phi_J \frac{\partial \Pi_J^\mu}{\partial \Pi_I^\nu} = \Phi_J \, \delta_{JI} \, \delta_\nu^\mu.$$

We thus end up with set of transformation rules

$$\pi_I^\mu = \frac{\partial F_2^\mu}{\partial \phi_I}, \qquad \Phi_I \, \delta_\nu^\mu = \frac{\partial F_2^\mu}{\partial \Pi_I^\nu}, \qquad \mathcal{H}' = \mathcal{H} + \left.\frac{\partial F_2^\alpha}{\partial x^\alpha}\right|_{\text{expl}}, \tag{11}$$

which is equivalent to the set (9) by virtue of the Legendre transformation (10) if the matrices $(\partial^2 F_1^\mu / \partial \phi_I \partial \Phi_J)$ are non-singular. From the second partial derivations of F_2^μ one immediately derives the symmetry relation

$$\frac{\partial \pi_I^\mu}{\partial \Pi_J^\nu} = \frac{\partial^2 F_2^\mu}{\partial \phi_I \partial \Pi_J^\nu} = \frac{\partial \Phi_J}{\partial \phi_I} \, \delta_\nu^\mu,$$

whose existence characterises the transformation to be canonical.

4. General inhomogeneous local gauge transformation

As a generalisation of the homogeneous local $\mathrm{U}(N)$ gauge transformation, we now treat the corresponding *inhomogeneous* $\mathrm{U}(N)$ gauge transformation for the particular case of an N-tuple of fields ϕ_I.

4.1. External gauge fields

We again consider a system consisting of an N-tuple $\boldsymbol{\phi}$ of complex fields ϕ_I with $I = 1, \ldots, N$, and $\overline{\boldsymbol{\phi}}$ its adjoint,

$$\boldsymbol{\phi} = (\,\phi\,)_1 \vdots \phi_N, \qquad \overline{\boldsymbol{\phi}} = \left(\overline{\phi}_1 \cdots \overline{\phi}_N\right).$$

A general inhomogeneous linear transformation may be expressed in terms of a complex matrix $U(x) = (u_{IJ}(x))$, $U^{\dagger}(x) = (\overline{u}_{IJ}(x))$ and a vector $\boldsymbol{\varphi}(x) = (\varphi_I(x))$ that generally depend explicitly on the independent variables, x^{μ}, as

$$\begin{aligned}
\boldsymbol{\Phi} &= U\,\boldsymbol{\phi} + \boldsymbol{\varphi}, \qquad & \overline{\boldsymbol{\Phi}} &= \overline{\boldsymbol{\phi}}\,U^{\dagger} + \overline{\boldsymbol{\varphi}} \\
\Phi_I &= u_{IJ}\,\phi_J + \varphi_I, \qquad & \overline{\Phi}_I &= \overline{\phi}_J\,\overline{u}_{JI} + \overline{\varphi}_I.
\end{aligned} \tag{12}$$

With this notation, ϕ_I stands for a set of $I = 1, \ldots, N$ complex fields ϕ_I. In other words, U is supposed to define an isomorphism within the space of the ϕ_I, hence to linearly map the ϕ_I into objects of the same type. The quantities $\varphi_I(x)$ have the dimension of the base fields ϕ_I and define a *local* shifting transformation of the Φ_I in iso-space.

The transformation (12) follows from a generating function that — corresponding to $\mathcal{H}$ — must be a real-valued function of the generally complex fields ϕ_I and their canonical conjugates, π_I^{μ},

$$\begin{aligned}
F_2^{\mu}(\boldsymbol{\phi}, \overline{\boldsymbol{\phi}}, \boldsymbol{\Pi}^{\mu}, \overline{\boldsymbol{\Pi}}^{\mu}, x) &= \overline{\boldsymbol{\Pi}}^{\mu}\left(U\,\boldsymbol{\phi} + \boldsymbol{\varphi}\right) + \left(\overline{\boldsymbol{\phi}}\,U^{\dagger} + \overline{\boldsymbol{\varphi}}\right)\boldsymbol{\Pi}^{\mu} \\
&= \overline{\Pi}_K^{\mu}\left(u_{KJ}\,\phi_J + \varphi_K\right) + \left(\overline{\phi}_K\,\overline{u}_{KJ} + \overline{\varphi}_J\right)\Pi_J^{\mu}.
\end{aligned} \tag{13}$$

According to Eqs. (11) the set of transformation rules follows as

$$\begin{aligned}
\overline{\pi}_I^{\mu} &= \frac{\partial F_2^{\mu}}{\partial \phi_I} = \overline{\Pi}_K^{\mu} u_{KJ}\delta_{JI}, \overline{\Phi}_I \delta_{\nu}^{\mu} = \frac{\partial F_2^{\mu}}{\partial \Pi_I^{\nu}} = \left(\overline{\phi}_K \overline{u}_{KJ} + \overline{\varphi}_J\right)\delta_{\nu}^{\mu}\delta_{JI} \\
\pi_I^{\mu} &= \frac{\partial F_2^{\mu}}{\partial \overline{\phi}_I} = \delta_{IK}\overline{u}_{KJ}\Pi_J^{\mu}, \Phi_I \delta_{\nu}^{\mu} = \frac{\partial F_2^{\mu}}{\partial \overline{\Pi}_I^{\nu}} = \delta_{\nu}^{\mu}\delta_{IK}\left(u_{KJ}\phi_J + \varphi_K\right).
\end{aligned}$$

The complete set of transformation rules and their inverses then read in component notation

$$\begin{aligned}
\Phi_I &= u_{IJ}\,\phi_J + \varphi_I, \qquad \overline{\Phi}_I = \overline{\phi}_J\,\overline{u}_{JI} + \overline{\varphi}_I, \Pi_I^{\mu} = u_{IJ}\,\pi_J^{\mu}, \overline{\Pi}_I^{\mu} = \overline{\pi}_J^{\mu}\,\overline{u}_{JI} \\
\phi_I &= \overline{u}_{IJ}\left(\Phi_J - \varphi_J\right), \overline{\phi}_I = \left(\overline{\Phi}_J - \overline{\varphi}_J\right)u_{JI}, \pi_I^{\mu} = \overline{u}_{IJ}\,\Pi_J^{\mu}, \overline{\pi}_I^{\mu} = \overline{\Pi}_J^{\mu} u_{JI}.
\end{aligned} \tag{14}$$

We restrict ourselves to transformations that preserve the contraction $\overline{\boldsymbol{\pi}}^{\alpha}\boldsymbol{\pi}_{\alpha}$

$$\begin{aligned}
\overline{\boldsymbol{\Pi}}^{\alpha}\boldsymbol{\Pi}_{\alpha} &= \overline{\boldsymbol{\pi}}^{\alpha}\,U^{\dagger}U\,\boldsymbol{\pi}_{\alpha} = \overline{\boldsymbol{\pi}}^{\alpha}\boldsymbol{\pi}_{\alpha} \qquad \Longrightarrow \qquad U^{\dagger}U = 1 = UU^{\dagger} \\
\overline{\Pi}_I^{\alpha}\Pi_{I\alpha} &= \overline{\pi}_J^{\alpha}\overline{u}_{JI}\,u_{IK}\pi_{K\alpha} = \overline{\pi}_K^{\alpha}\pi_{K\alpha} \Longrightarrow \quad \overline{u}_{JI}\,u_{IK} = \delta_{JK} = u_{JI}\,\overline{u}_{IK}.
\end{aligned}$$

This means that $U^{\dagger} = U^{-1}$, hence that the matrix U is supposed to be *unitary*. As a unitary matrix, $U(x)$ is a member of the unitary group U(N)

$$U^{\dagger}(x) = U^{-1}(x), \qquad |\det U(x)| = 1.$$

For $\det U(x) = +1$, the matrix $U(x)$ is a member of the special group SU(N).

We require the Hamiltonian density $\mathcal{H}$ to be *form-invariant* under the *global* gauge transformation (12), which is given for $U, \boldsymbol{\varphi} = \mathrm{const.}$, hence for all u_{IJ}, φ_I *not* depending on the independent variables, x^μ. Generally, if $U = U(x)$, $\boldsymbol{\varphi} = \boldsymbol{\varphi}(x)$, then the transformation (14) is referred to as a *local* gauge transformation. The transformation rule for the Hamiltonian is then determined by the explicitly x^μ-dependent terms of the generating function F_2^μ according to

$$\begin{aligned}\mathcal{H}' - \mathcal{H} = \left.\frac{\partial F_2^\alpha}{\partial x^\alpha}\right|_{\mathrm{expl}} &= \overline{\Pi}_I^\alpha \left(\frac{\partial u_{IJ}}{\partial x^\alpha}\,\phi_J + \frac{\partial \varphi_I}{\partial x^\alpha}\right) + \left(\overline{\phi}_I \frac{\partial \overline{u}_{IJ}}{\partial x^\alpha} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha}\right)\Pi_J^\alpha \\ &= \overline{\pi}_K^\alpha\,\overline{u}_{KI}\left(\frac{\partial u_{IJ}}{\partial x^\alpha}\phi_J + \frac{\partial \varphi_I}{\partial x^\alpha}\right) + \left(\overline{\phi}_I \frac{\partial \overline{u}_{IJ}}{\partial x^\alpha} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha}\right) u_{JK}\pi_K^\alpha \\ &= \left(\overline{\pi}_K^\alpha\,\phi_J - \overline{\phi}_K \pi_J^\alpha\right)\overline{u}_{KI}\frac{\partial u_{IJ}}{\partial x^\alpha} + \overline{\pi}_I^\alpha \overline{u}_{IJ}\frac{\partial \varphi_J}{\partial x^\alpha} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha} u_{JI}\pi_I^\alpha. \end{aligned} \tag{15}$$

In the last step, the identity

$$\frac{\partial \overline{u}_{JI}}{\partial x^\mu}\,u_{IK} + \overline{u}_{JI}\,\frac{\partial u_{IK}}{\partial x^\mu} = 0$$

was inserted. If we want to set up a Hamiltonian $\mathcal{H}_1$ that is *form-invariant* under the *local*, hence x^μ-dependent transformation generated by (13), then we must compensate the additional terms (15) that emerge from the explicit x^μ-dependence of the generating function (13). The only way to achieve this is to *adjoin* the Hamiltonian $\mathcal{H}$ of our system with terms that correspond to (15) with regard to their dependence on the canonical variables, $\boldsymbol{\phi}, \overline{\boldsymbol{\phi}}, \boldsymbol{\pi}^\mu, \overline{\boldsymbol{\pi}}^\mu$. With a *unitary* matrix U, the u_{IJ}-dependent terms in Eq. (15) are *skew-Hermitian*,

$$\overline{\overline{u}_{KI}\,\frac{\partial u_{IJ}}{\partial x^\mu}} = \frac{\partial \overline{u}_{JI}}{\partial x^\mu}\,u_{IK} = -\overline{u}_{JI}\,\frac{\partial u_{IK}}{\partial x^\mu}, \qquad \overline{\frac{\partial u_{KI}}{\partial x^\mu}\,\overline{u}_{IJ}} = u_{JI}\frac{\partial \overline{u}_{IK}}{\partial x^\mu} = -\frac{\partial u_{JI}}{\partial x^\mu}\overline{u}_{IK},$$

or in matrix notation

$$\left(U^\dagger \frac{\partial U}{\partial x^\mu}\right)^\dagger = \frac{\partial U^\dagger}{\partial x^\mu}U = -U^\dagger \frac{\partial U}{\partial x^\mu}, \qquad \left(\frac{\partial U}{\partial x^\mu}U^\dagger\right)^\dagger = U\frac{\partial U^\dagger}{\partial x^\mu} = -\frac{\partial U}{\partial x^\mu}U^\dagger.$$

The $\overline{u}_{KI}\partial u_{IJ}/\partial x^\mu$-dependent terms in Eq. (15) can thus be compensated by a *Hermitian* matrix $(\boldsymbol{a}_{KJ})$ of "4-vector gauge fields", with each off-diagonal matrix element, $\boldsymbol{a}_{KJ},\ K \neq J$, a complex 4-vector field with components $a_{KJ\mu},\ \mu = 0, \ldots, 3$

$$\overline{u}_{KI}\frac{\partial u_{IJ}}{\partial x^\mu} \quad \leftrightarrow \quad a_{KJ\mu}, \qquad a_{KJ\mu} = \overline{a}_{KJ\mu} = a^*_{JK\mu}.$$

Correspondingly, the term proportional to $\overline{u}_{IJ}\partial\varphi_J/\partial x^\mu$ is compensated by the μ-components $M_{IJ}b_{J\mu}$ of a vector $M_{IJ}\,\boldsymbol{b}_J$ of 4-vector gauge fields,

$$\overline{u}_{IJ}\frac{\partial \varphi_J}{\partial x^\mu} \quad \leftrightarrow \quad M_{IJ}b_{J\mu}, \qquad \frac{\partial \overline{\varphi}_J}{\partial x^\mu}u_{JI} \quad \leftrightarrow \quad \overline{b}_{J\mu}M_{IJ}.$$

The term proportional to $\partial\overline{\varphi}_J/\partial x\,u_{JI}$ is then compensated by the adjoint vector $\overline{\boldsymbol{b}}_J M_{IJ}$. The dimension of the constant real matrix M is $[M] = L^{-1}$ and thus has the natural dimension of mass. The given system Hamiltonian $\mathcal{H}$ must be amended by a Hamiltonian $\mathcal{H}_\mathrm{a}$ of the form

$$\mathcal{H}_1 = \mathcal{H} + \mathcal{H}_\mathrm{a}, \quad \mathcal{H}_\mathrm{a} = \mathrm{i}g\left(\overline{\pi}_K^\alpha \phi_J - \overline{\phi}_K \pi_J^\alpha\right)a_{KJ\alpha} + \overline{\pi}_I^\alpha M_{IJ}b_{J\alpha} + \overline{b}_{J\alpha}M_{IJ}\pi_I^\alpha \tag{16}$$

in order for $\mathcal{H}_1$ to be *form-invariant* under the canonical transformation that is defined by the explicitly x^μ-dependent generating function from Eq. (13). With a real coupling constant g, the “gauge Hamiltonian” $\mathcal{H}_\mathrm{a}$ is thus real. Submitting the amended Hamiltonian $\mathcal{H}_1$ to the canonical transformation generated by Eq. (13), the new Hamiltonian $\mathcal{H}_1'$ emerges as

$$\begin{aligned}\mathcal{H}_1' &= \mathcal{H}_1 + \left.\frac{\partial F_2^\alpha}{\partial x^\alpha}\right|_{\mathrm{expl}} = \mathcal{H} + \mathcal{H}_\mathrm{a} + \left.\frac{\partial F_2^\alpha}{\partial x^\alpha}\right|_{\mathrm{expl}} \\ &= \mathcal{H} + \left(\overline{\pi}_K^\alpha \phi_J - \overline{\phi}_K \pi_J^\alpha\right)\left(\mathrm{i}g\, a_{KJ\alpha} + \overline{u}_{KI}\frac{\partial u_{IJ}}{\partial x^\alpha}\right) \\ &\quad + \overline{\pi}_I^\alpha\left(M_{IJ} b_{J\alpha} + \overline{u}_{IJ}\frac{\partial \varphi_J}{\partial x^\alpha}\right) + \left(\overline{b}_{J\alpha} M_{IJ} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha} u_{JI}\right)\pi_I^\alpha \\ &\stackrel{!}{=} \mathcal{H}' + \mathrm{i}g\left(\overline{\Pi}_K^\alpha \Phi_J - \overline{\Phi}_K \Pi_J^\alpha\right) A_{KJ\alpha} + \overline{\Pi}_I^\alpha M_{IJ} B_{J\alpha} + \overline{B}_{J\alpha} M_{IJ} \Pi_I^\alpha,\end{aligned}$$

with the $A_{IJ\mu}$ and $B_{I\mu}$ defining the gauge field components of the transformed system. The *form* of the system Hamiltonian $\mathcal{H}_1$ is thus maintained under the canonical transformation,

$$\mathcal{H}_1' = \mathcal{H}' + \mathcal{H}_\mathrm{a}', \qquad \mathcal{H}_\mathrm{a}' = \mathrm{i}g\left(\overline{\Pi}_K^\alpha \Phi_J - \overline{\Phi}_K \Pi_J^\alpha\right) A_{KJ\alpha} + \overline{\Pi}_I^\alpha M_{IJ} B_{J\alpha} + \overline{B}_{J\alpha} M_{IJ} \Pi_I^\alpha, \tag{17}$$

provided that the given system Hamiltonian $\mathcal{H}$ is form-invariant under the corresponding *global* gauge transformation (14). In other words, we suppose the given system Hamiltonian $\mathcal{H}(\boldsymbol{\phi}, \overline{\boldsymbol{\phi}}, \boldsymbol{\pi}^\mu, \overline{\boldsymbol{\pi}}^\mu, x)$ to remain form-invariant if it is expressed in terms of the transformed fields,

$$\mathcal{H}'(\boldsymbol{\Phi}, \overline{\boldsymbol{\Phi}}, \boldsymbol{\Pi}^\mu, \overline{\boldsymbol{\Pi}}^\mu, x) \stackrel{\text{global GT}}{=} \mathcal{H}(\boldsymbol{\phi}, \overline{\boldsymbol{\phi}}, \boldsymbol{\pi}^\mu, \overline{\boldsymbol{\pi}}^\mu, x).$$

The gauge fields must then satisfy the condition

$$\begin{aligned}&\mathrm{i}g\left(\overline{\Pi}_K^\alpha \Phi_J - \overline{\Phi}_K \Pi_J^\alpha\right) A_{KJ\alpha} + \overline{\Pi}_I^\alpha M_{IJ} B_{J\alpha} + \overline{B}_{J\alpha} M_{IJ} \Pi_I^\alpha \\ &= \left(\overline{\pi}_K^\alpha \phi_J - \overline{\phi}_K \pi_J^\alpha\right)\left(\mathrm{i}g\, a_{KJ\alpha} + \overline{u}_{KI}\frac{\partial u_{IJ}}{\partial x^\alpha}\right) \\ &\quad + \overline{\pi}_I^\alpha\left(M_{IJ} b_{J\alpha} + \overline{u}_{IJ}\frac{\partial \varphi_J}{\partial x^\alpha}\right) + \left(\overline{b}_{J\alpha} M_{IJ} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha} u_{JI}\right)\pi_I^\alpha,\end{aligned}$$

which yields with Eqs. (14) the following inhomogeneous transformation rules for the gauge fields $\boldsymbol{a}_{KJ}$, $\boldsymbol{b}_J$, and $\overline{\boldsymbol{b}}_J$

$$\begin{aligned}A_{KJ\mu} &= u_{KL}\, a_{LI\mu}\, \overline{u}_{IJ} + \frac{1}{\mathrm{i}g}\frac{\partial u_{KI}}{\partial x^\mu}\overline{u}_{IJ} \\ B_{J\mu} &= \tilde{M}_{JI}\left(u_{IK} M_{KL} b_{L\mu} - \mathrm{i}g\, A_{IK\mu}\varphi_K + \frac{\partial \varphi_I}{\partial x^\mu}\right) \\ \overline{B}_{J\mu} &= \left(\overline{b}_{L\mu} M_{KL} \overline{u}_{KI} + \mathrm{i}g\, \overline{\varphi}_K A_{KI\mu} + \frac{\partial \overline{\varphi}_I}{\partial x^\mu}\right)\tilde{M}_{JI}.\end{aligned} \tag{18}$$

Herein, $\tilde{M}$ denotes the inverse matrix of M, hence $\tilde{M}_{KJ} M_{JI} = M_{KJ}\tilde{M}_{JI} = \delta_{KI}$. We observe that for any type of canonical field variables ϕ_I and for any Hamiltonian system $\mathcal{H}$, the transformation of both the matrix $\boldsymbol{a}_{IJ}$ as well as the vector $\boldsymbol{b}_I$ of 4-vector gauge fields is uniquely determined according to Eq. (18) by the unitary matrix $U(x)$ and the translation vector $\boldsymbol{\varphi}(x)$ that determine the *local* transformation of the N base fields $\boldsymbol{\phi}$. In a more concise

matrix notation, Eqs. (18) are

$$\begin{aligned}\boldsymbol{A}_\mu &= U\,\boldsymbol{a}_\mu\,U^\dagger + \frac{1}{\mathrm{i}g}\,\frac{\partial U}{\partial x^\mu}\,U^\dagger \\ M\boldsymbol{B}_\mu &= UM\,\boldsymbol{b}_\mu - \mathrm{i}g\,\boldsymbol{A}_\mu\boldsymbol{\varphi} + \frac{\partial\boldsymbol{\varphi}}{\partial x^\mu} \\ \overline{\boldsymbol{B}}_\mu M^T &= \overline{\boldsymbol{b}}_\mu M^T\,U^\dagger + \mathrm{i}g\,\overline{\boldsymbol{\varphi}}\,\boldsymbol{A}_\mu + \frac{\partial\overline{\boldsymbol{\varphi}}}{\partial x^\mu}.\end{aligned} \tag{19}$$

4.2. Including the gauge field dynamics

With the knowledge of the required transformation rule for the gauge fields from Eq. (18), it is now possible to redefine the generating function (13) to also describe the gauge field transformation. This simultaneously defines the transformation of the canonical conjugates, $p_{JK}^{\mu\nu}$ and $q_J^{\mu\nu}$, of the gauge fields $a_{JK\mu}$ and $b_{J\mu}$, respectively. Furthermore, the redefined generating function yields additional terms in the transformation rule for the Hamiltonian. Of course, in order for the Hamiltonian to be invariant under local gauge transformations, the additional terms must be invariant as well. The transformation rules for the base fields ϕ_I and the gauge fields $\boldsymbol{a}_{IJ}, \boldsymbol{b}_I$ (Eq. (18)) can be regarded as a canonical transformation that emerges from an explicitly x^μ-dependent and real-valued generating function vector of type $F_2^\mu = F_2^\mu(\boldsymbol{\phi}, \overline{\boldsymbol{\phi}}, \boldsymbol{\Pi}, \overline{\boldsymbol{\Pi}}, \boldsymbol{a}, \boldsymbol{P}, \boldsymbol{b}, \bar{\boldsymbol{b}}, \boldsymbol{Q}, \overline{\boldsymbol{Q}}, x)$,

$$\begin{aligned}F_2^\mu &= \overline{\Pi}_K^\mu\left(u_{KJ}\,\phi_J + \varphi_K\right) + \left(\overline{\phi}_K\,\overline{u}_{KJ} + \overline{\varphi}_J\right)\Pi_J^\mu \\ &\quad + \left(P_{JK}^{\alpha\mu} + \mathrm{i}g\,\tilde{M}_{LJ}Q_L^{\alpha\mu}\overline{\varphi}_K - \mathrm{i}g\,\varphi_J\overline{Q}_L^{\alpha\mu}\tilde{M}_{LK}\right)\left(u_{KN}\,a_{NI\alpha}\,\overline{u}_{IJ} + \frac{1}{\mathrm{i}g}\frac{\partial u_{KI}}{\partial x^\alpha}\,\overline{u}_{IJ}\right) \\ &\quad + \overline{Q}_L^{\alpha\mu}\tilde{M}_{LK}\left(u_{KI}M_{IJ}b_{J\alpha} + \frac{\partial\varphi_K}{\partial x^\alpha}\right) + \left(\overline{b}_{K\alpha}M_{IK}\overline{u}_{IJ} + \frac{\partial\overline{\varphi}_J}{\partial x^\alpha}\right)\tilde{M}_{LJ}Q_L^{\alpha\mu}.\end{aligned} \tag{20}$$

With the first line of (20) matching Eq. (13), the transformation rules for canonical variables $\boldsymbol{\phi}, \overline{\boldsymbol{\phi}}$ and their conjugates, $\boldsymbol{\pi}^\mu, \overline{\boldsymbol{\pi}}^\mu$, agree with those from Eqs. (14). The rule for the gauge fields $A_{KJ\alpha}$, $B_{K\alpha}$, and $\overline{B}_{K\alpha}$ emerge as

$$\begin{aligned}A_{KJ\alpha}\,\delta_\nu^\mu &= \frac{\partial F_2^\mu}{\partial P_{JK}^{\alpha\nu}} = \delta_\nu^\mu\left(u_{KN}\,a_{NI\alpha}\,\overline{u}_{IJ} + \frac{1}{\mathrm{i}g}\frac{\partial u_{KI}}{\partial x^\alpha}\overline{u}_{IJ}\right) \\ B_{L\alpha}\,\delta_\nu^\mu &= \frac{\partial F_2^\mu}{\partial\overline{Q}_L^{\alpha\nu}} = \delta_\nu^\mu\tilde{M}_{LK}\left[u_{KI}M_{IJ}b_{J\alpha} + \frac{\partial\varphi_K}{\partial x^\alpha} - \left(\mathrm{i}g\,u_{KN}\,a_{NI\alpha}\,\overline{u}_{IJ} + \frac{\partial u_{KI}}{\partial x^\alpha}\overline{u}_{IJ}\right)\varphi_J\right] \\ &= \delta_\nu^\mu\tilde{M}_{LK}\left(u_{KI}M_{IJ}b_{J\alpha} + \frac{\partial\varphi_K}{\partial x^\alpha} - \mathrm{i}g\,A_{KJ\alpha}\varphi_J\right) \\ \overline{B}_{L\alpha}\,\delta_\nu^\mu &= \frac{\partial F_2^\mu}{\partial Q_L^{\alpha\nu}} = \delta_\nu^\mu\left[\overline{b}_{K\alpha}M_{IK}\overline{u}_{IJ} + \frac{\partial\overline{\varphi}_J}{\partial x^\alpha} + \overline{\varphi}_K\left(\mathrm{i}g\,u_{KN}\,a_{NI\alpha}\,\overline{u}_{IJ} + \frac{\partial u_{KI}}{\partial x^\alpha}\overline{u}_{IJ}\right)\right]\tilde{M}_{LJ} \\ &= \delta_\nu^\mu\left(\overline{b}_{K\alpha}M_{IK}\overline{u}_{IJ} + \frac{\partial\overline{\varphi}_J}{\partial x^\alpha} + \mathrm{i}g\,\overline{\varphi}_K\,A_{KJ\alpha}\right)\tilde{M}_{LJ},\end{aligned}$$

which obviously coincide with Eqs. (18) as the generating function (20) was devised accordingly. The transformation of the conjugate momentum fields is obtained from the generating function (20) as

$$q_J^{\nu\mu} = \frac{\partial F_2^\mu}{\partial\bar{b}_{J\nu}} = M_{IJ}\,\overline{u}_{IK}\,\tilde{M}_{LK}\,Q_L^{\nu\mu}, \qquad \tilde{M}_{KJ}Q_K^{\nu\mu} = u_{JI}\,\tilde{M}_{KI}\,q_K^{\nu\mu}$$

$$\overline{q}_J^{\nu\mu} = \frac{\partial F_2^\mu}{\partial b_{J\nu}} \quad = \overline{Q}_L^{\nu\mu} \tilde{M}_{LK}\, u_{KI}\, M_{IJ}, \qquad \overline{Q}_K^{\nu\mu} \tilde{M}_{KJ} = \overline{q}_K^{\nu\mu} \tilde{M}_{KI}\, \overline{u}_{IJ} \tag{21}$$

$$\begin{aligned}
p_{IN}^{\nu\mu} = \frac{\partial F_2^\mu}{\partial a_{NI\nu}} &= \overline{u}_{IJ} \left(P_{JK}^{\nu\mu} + \mathrm{i}g\, \tilde{M}_{LJ} Q_L^{\nu\mu}\, \overline{\varphi}_K - \mathrm{i}g\, \varphi_J\, \overline{Q}_L^{\nu\mu} \tilde{M}_{LK} \right) u_{KN} \\
&= \overline{u}_{IJ} \left(P_{JK}^{\nu\mu} + \mathrm{i}g\, \tilde{M}_{LJ} Q_L^{\nu\mu}\, \overline{\Phi}_K - \mathrm{i}g\, \Phi_J\, \overline{Q}_L^{\nu\mu} \tilde{M}_{LK} \right) u_{KN} \\
&\qquad - \mathrm{i}g\, \tilde{M}_{LI} q_L^{\nu\mu}\, \overline{\phi}_N \;+\; \mathrm{i}g\, \phi_I\, \overline{q}_L^{\nu\mu} \tilde{M}_{LN}.
\end{aligned}$$

Thus, the expression

$$\begin{aligned}
&p_{IN}^{\nu\mu} + \mathrm{i}g\, \tilde{M}_{LI} q_L^{\nu\mu}\, \overline{\phi}_N - \mathrm{i}g\, \phi_I\, \overline{q}_L^{\nu\mu} \tilde{M}_{LN} \\
&= \overline{u}_{IJ} \left(P_{JK}^{\nu\mu} + \mathrm{i}g\, \tilde{M}_{LJ} Q_L^{\nu\mu}\, \overline{\Phi}_K - \mathrm{i}g\, \Phi_J\, \overline{Q}_L^{\nu\mu} \tilde{M}_{LK} \right) u_{KN}
\end{aligned} \tag{22}$$

transforms *homogeneously* under the gauge transformation generated by Eq. (20). The same homogeneous transformation law holds for the expression

$$\begin{aligned}
f_{IJ\mu\nu} &= \frac{\partial a_{IJ\nu}}{\partial x^\mu} - \frac{\partial a_{IJ\mu}}{\partial x^\nu} + \mathrm{i}g\, \left(a_{IK\nu} a_{KJ\mu} - a_{IK\mu} a_{KJ\nu} \right) \\
&= \overline{u}_{IK}\, F_{KL\mu\nu}\, u_{LJ} \\
F_{IJ\mu\nu} &= \frac{\partial A_{IJ\nu}}{\partial x^\mu} - \frac{\partial A_{IJ\mu}}{\partial x^\nu} + \mathrm{i}g\, \left(A_{IK\nu} A_{KJ\mu} - A_{IK\mu} A_{KJ\nu} \right),
\end{aligned} \tag{23}$$

which directly follows from the transformation rule (18) for the gauge fields $a_{IJ\mu}$. Making use of the initially defined mapping of the base fields (12), the transformation rule (18) for the gauge fields $b_{K\mu}, \overline{b}_{K\mu}$ is converted into

$$\begin{aligned}
&\frac{\partial \Phi_J}{\partial x^\mu} - \mathrm{i}g\, A_{JK\mu} \Phi_K - M_{JK} B_{K\mu} = u_{JL} \left(\frac{\partial \phi_L}{\partial x^\mu} - \mathrm{i}g\, a_{LK\mu} \phi_K - M_{LK} b_{K\mu} \right) \\
&\frac{\partial \overline{\Phi}_J}{\partial x^\mu} + \mathrm{i}g\, \overline{\Phi}_K A_{KJ\mu} - \overline{B}_{K\mu} M_{JK} = \left(\frac{\partial \overline{\phi}_L}{\partial x^\mu} + \mathrm{i}g\, \overline{\phi}_K a_{KL\mu} - \overline{b}_{K\mu} M_{LK} \right) \overline{u}_{LJ}.
\end{aligned} \tag{24}$$

The above transformation rules can also be expressed more clearly in matrix notation

$$\begin{aligned}
\boldsymbol{q}^{\nu\mu} &= M^T U^\dagger \tilde{M}^T \boldsymbol{Q}^{\nu\mu}, \qquad \tilde{M}^T \boldsymbol{Q}^{\nu\mu} = U \tilde{M}^T \boldsymbol{q}^{\nu\mu} \\
\overline{\boldsymbol{q}}^{\nu\mu} &= \overline{\boldsymbol{Q}}^{\nu\mu} \tilde{M}\, U\, M, \qquad \overline{\boldsymbol{Q}}^{\nu\mu} \tilde{M} = \overline{\boldsymbol{q}}^{\nu\mu} \tilde{M}\, U^\dagger \\
\boldsymbol{p}^{\nu\mu} &= U^\dagger \left(\boldsymbol{P}^{\nu\mu} + \mathrm{i}g\, \tilde{M}^T \boldsymbol{Q}^{\nu\mu} \otimes \overline{\boldsymbol{\varphi}} - \mathrm{i}g\, \boldsymbol{\varphi} \otimes \overline{\boldsymbol{Q}}^{\nu\mu} \tilde{M} \right) U \\
\boldsymbol{f}_{\mu\nu} &= U^\dagger\, \boldsymbol{F}_{\mu\nu}\, U, \qquad \boldsymbol{f}_{\mu\nu} = \frac{\partial \boldsymbol{a}_\nu}{\partial x^\mu} - \frac{\partial \boldsymbol{a}_\mu}{\partial x^\nu} + \mathrm{i}g\, (\boldsymbol{a}_\nu \boldsymbol{a}_\mu - \boldsymbol{a}_\mu \boldsymbol{a}_\nu)
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
\frac{\partial \boldsymbol{\Phi}}{\partial x^\mu} - \mathrm{i}g\, \boldsymbol{A}_\mu \boldsymbol{\Phi} - M \boldsymbol{B}_\mu \qquad &= U \left(\frac{\partial \boldsymbol{\phi}}{\partial x^\mu} - \mathrm{i}g\, \boldsymbol{a}_\mu \boldsymbol{\phi} - M\, \boldsymbol{b}_\mu \right) \\
\frac{\partial \overline{\boldsymbol{\Phi}}}{\partial x^\mu} + \mathrm{i}g\, \overline{\boldsymbol{\Phi}} \boldsymbol{A}_\mu - \overline{\boldsymbol{B}}_\mu M^T \qquad &= \left(\frac{\partial \overline{\boldsymbol{\phi}}}{\partial x^\mu} + \mathrm{i}g\, \overline{\boldsymbol{\phi}}\, \boldsymbol{a}_\mu - \overline{\boldsymbol{b}}_\mu M^T \right) U^\dagger \\
\boldsymbol{P}^{\nu\mu} + \mathrm{i}g\, \tilde{M}^T \boldsymbol{Q}^{\nu\mu} \otimes \overline{\boldsymbol{\Phi}} - \mathrm{i}g\, \boldsymbol{\Phi} \otimes \overline{\boldsymbol{Q}}^{\nu\mu} \tilde{M} &= U \left(\boldsymbol{p}^{\nu\mu} + \mathrm{i}g\, \tilde{M}^T \boldsymbol{q}^{\nu\mu} \otimes \overline{\boldsymbol{\phi}} - \mathrm{i}g\, \boldsymbol{\phi} \otimes \overline{\boldsymbol{q}}^{\nu\mu} \tilde{M} \right) U^\dagger.
\end{aligned}$$

Equation (24) can be regarded as an “extended minimum coupling rule,” with the respective third terms arising from the inhomogeneous part of the gauge transformation.

It remains to work out the difference of the Hamiltonians that are submitted to the canonical transformation generated by (20). Hence, according to the general rule from Eq. (11), we must calculate the divergence of the explicitly x^μ-dependent terms of F_2^μ

$$
\begin{aligned}
\left.\frac{\partial F_2^\alpha}{\partial x^\alpha}\right|_{\mathrm{expl}} &= \overline{\Pi}_K^\alpha \left(\frac{\partial u_{KJ}}{\partial x^\alpha}\,\phi_J + \frac{\partial \varphi_K}{\partial x^\alpha}\right) + \left(\overline{\phi}_K\,\frac{\partial \overline{u}_{KJ}}{\partial x^\alpha} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha}\right)\Pi_J^\alpha \\
&+ \left(P_{JK}^{\alpha\beta} + \mathrm{i}g\,\tilde{M}_{LJ}Q_L^{\alpha\beta}\,\overline{\varphi}_K - \mathrm{i}g\,\varphi_J\,\overline{Q}_L^{\alpha\beta}\tilde{M}_{LK}\right) \\
&\quad\cdot\left(\frac{\partial u_{KN}}{\partial x^\beta}a_{NI\alpha}\overline{u}_{IJ} + u_{KN}a_{NI\alpha}\frac{\partial \overline{u}_{IJ}}{\partial x^\beta} + \frac{1}{\mathrm{i}g}\frac{\partial u_{KI}}{\partial x^\alpha}\frac{\partial \overline{u}_{IJ}}{\partial x^\beta} + \frac{1}{\mathrm{i}g}\frac{\partial^2 u_{KI}}{\partial x^\alpha\partial x^\beta}\overline{u}_{IJ}\right) \\
&+ \left(\tilde{M}_{LJ}Q_L^{\alpha\beta}\frac{\partial \overline{\varphi}_K}{\partial x^\beta} - \frac{\partial \varphi_J}{\partial x^\beta}\,\overline{Q}_L^{\alpha\beta}\tilde{M}_{LK}\right)\left(\mathrm{i}g\,u_{KN}\,a_{NI\alpha}\,\overline{u}_{IJ} + \frac{\partial u_{KI}}{\partial x^\alpha}\,\overline{u}_{IJ}\right) \\
&+ \overline{Q}_L^{\alpha\beta}\tilde{M}_{LK}\left(\frac{\partial u_{KI}}{\partial x^\beta}M_{IJ}b_{J\alpha} + \frac{\partial^2 \varphi_K}{\partial x^\alpha\partial x^\beta}\right) + \left(\overline{b}_{K\alpha}M_{IK}\frac{\partial \overline{u}_{IJ}}{\partial x^\beta} + \frac{\partial^2 \overline{\varphi}_J}{\partial x^\alpha\partial x^\beta}\right)\tilde{M}_{LJ}Q_L^{\alpha\beta}. \qquad (26)
\end{aligned}
$$

We are now going to replace all u_{IJ}- and φ_K-dependencies in (26) by canonical variables making use of the canonical transformation rules. To this end, the terms of Eq. (26) are split into three blocks. The $\mathbf{\Pi}$-dependent terms of can be converted this way by means of the transformation rules (14) and (18)

$$
\begin{aligned}
&\overline{\Pi}_K^\alpha \left(\frac{\partial u_{KJ}}{\partial x^\alpha}\,\phi_J + \frac{\partial \varphi_K}{\partial x^\alpha}\right) + \left(\overline{\phi}_K\,\frac{\partial \overline{u}_{KJ}}{\partial x^\alpha} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha}\right)\Pi_J^\alpha \\
&= \overline{\Pi}_K^\alpha \left(\frac{\partial u_{KJ}}{\partial x^\alpha}\,\overline{u}_{JI}(\Phi_I - \varphi_I) + \frac{\partial \varphi_K}{\partial x^\alpha}\right) + \left(\left(\overline{\Phi}_I - \overline{\varphi}_I\right)u_{IK}\frac{\partial \overline{u}_{KJ}}{\partial x^\alpha} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha}\right)\Pi_J^\alpha \\
&= \mathrm{i}g\left(\overline{\Pi}_K^\alpha\Phi_J - \overline{\Phi}_K\Pi_J^\alpha\right)A_{KJ\alpha} + \overline{\Pi}_K^\alpha M_{KJ}B_{J\alpha} + \overline{B}_{K\alpha}M_{JK}\Pi_J^\alpha \\
&\quad - \mathrm{i}g\left(\overline{\pi}_K^\alpha\phi_J - \overline{\phi}_K\pi_J^\alpha\right)a_{KJ\alpha} - \left(\overline{\pi}_K^\alpha M_{KJ}b_{J\alpha} + \overline{b}_{K\alpha}M_{JK}\pi_J^\alpha\right). \qquad (27)
\end{aligned}
$$

The second derivative terms in Eq. (26) are *symmetric* in the indices α and β. If we split $P_{JK}^{\alpha\beta}$ and $Q_J^{\alpha\beta}$ into a symmetric $P_{JK}^{(\alpha\beta)}, Q_J^{(\alpha\beta)}$ and a skew-symmetric parts $P_{JK}^{[\alpha\beta]}, P_J^{[\alpha\beta]}$ in α and β

$$
\begin{aligned}
P_{JK}^{\alpha\beta} &= P_{JK}^{(\alpha\beta)} + P_{JK}^{[\alpha\beta]}, &\quad P_{JK}^{[\alpha\beta]} &= \tfrac{1}{2}\left(P_{JK}^{\alpha\beta} - P_{JK}^{\beta\alpha}\right), &\quad P_{JK}^{(\alpha\beta)} &= \tfrac{1}{2}\left(P_{JK}^{\alpha\beta} + P_{JK}^{\beta\alpha}\right) \\
Q_J^{\alpha\beta} &= Q_J^{(\alpha\beta)} + Q_J^{[\alpha\beta]}, &\quad Q_J^{[\alpha\beta]} &= \tfrac{1}{2}\left(Q_J^{\alpha\beta} - Q_J^{\beta\alpha}\right), &\quad Q_J^{(\alpha\beta)} &= \tfrac{1}{2}\left(Q_J^{\alpha\beta} + Q_J^{\beta\alpha}\right),
\end{aligned}
$$

then the second derivative terms in Eq. (26) vanish for $P_{JK}^{[\alpha\beta]}$ and $Q_J^{[\alpha\beta]}$,

$$
P_{JK}^{[\alpha\beta]}\frac{\partial^2 u_{KI}}{\partial x^\alpha\partial x^\beta} = 0, \qquad \frac{\partial^2 \overline{\varphi}_J}{\partial x^\alpha\partial x^\beta}Q_J^{[\alpha\beta]} = 0, \qquad \overline{Q}_K^{[\alpha\beta]}\frac{\partial^2 \varphi_K}{\partial x^\alpha\partial x^\beta} = 0.
$$

By inserting the transformation rules for the gauge fields from Eqs. (18), the remaining terms of (26) for the skew-symmetric part of $P_{JK}^{\alpha\beta}$ are converted into

$$
\begin{aligned}
&\left(P_{JK}^{[\alpha\beta]} + \mathrm{i}g\,\tilde{M}_{LJ}Q_L^{[\alpha\beta]}\,\overline{\varphi}_K - \mathrm{i}g\,\varphi_J\,\overline{Q}_L^{[\alpha\beta]}\tilde{M}_{LK}\right) \\
&\quad\cdot\left(\frac{\partial u_{KN}}{\partial x^\beta}a_{NI\alpha}\overline{u}_{IJ} + u_{KN}a_{NI\alpha}\frac{\partial \overline{u}_{IJ}}{\partial x^\beta} + \frac{1}{\mathrm{i}g}\frac{\partial u_{KI}}{\partial x^\alpha}\frac{\partial \overline{u}_{IJ}}{\partial x^\beta}\right) \\
&\quad+ \left(\tilde{M}_{LJ}Q_L^{[\alpha\beta]}\,\frac{\partial \overline{\varphi}_K}{\partial x^\beta} - \frac{\partial \varphi_J}{\partial x^\beta}\,\overline{Q}_L^{[\alpha\beta]}\tilde{M}_{LK}\right)\mathrm{i}g\,A_{KJ\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&+\overline{Q}_L^{[\alpha\beta]}\tilde{M}_{LK}\frac{\partial u_{KI}}{\partial x^\beta}M_{IJ}b_{J\alpha}+\overline{b}_{J\alpha}M_{IJ}\frac{\partial\overline{u}_{IK}}{\partial x^\beta}\tilde{M}_{LK}Q_L^{[\alpha\beta]}\\
=&-\tfrac{1}{2}\mathrm{i}g\,P_{JK}^{\alpha\beta}\left(A_{KI\alpha}A_{IJ\beta}-A_{KI\beta}A_{IJ\alpha}\right)\\
&+\tfrac{1}{2}\mathrm{i}g\left(\overline{B}_{J\beta}M_{KJ}A_{KI\alpha}\tilde{M}_{IL}-\overline{B}_{J\alpha}M_{KJ}A_{KI\beta}\tilde{M}_{IL}\right)Q_L^{\alpha\beta}\\
&-\tfrac{1}{2}\mathrm{i}g\,\overline{Q}_L^{\alpha\beta}\left(\tilde{M}_{LI}A_{IK\alpha}M_{KJ}B_{J\beta}-\tilde{M}_{LI}A_{IK\beta}M_{KJ}B_{J\alpha}\right)\\
&+\tfrac{1}{2}\mathrm{i}g\,p_{JK}^{\alpha\beta}\left(a_{KI\alpha}a_{IJ\beta}-a_{KI\beta}a_{IJ\alpha}\right)\\
&-\tfrac{1}{2}\mathrm{i}g\left(\overline{b}_{J\beta}M_{KJ}a_{KI\alpha}\tilde{M}_{IL}-\overline{b}_{J\alpha}M_{KJ}a_{KI\beta}\tilde{M}_{LI}\right)q_L^{\alpha\beta}\\
&+\tfrac{1}{2}\mathrm{i}g\,\overline{q}_L^{\alpha\beta}\left(\tilde{M}_{LI}a_{IK\alpha}M_{KJ}b_{J\beta}-\tilde{M}_{LI}a_{IK\beta}M_{KJ}b_{J\alpha}\right).
\end{aligned}
\tag{28}
$$

For the symmetric parts of $P_{JK}^{\alpha\beta}$ and $Q_J^{\alpha\beta}$, we obtain

$$
\begin{aligned}
&\left(P_{JK}^{(\alpha\beta)}+\mathrm{i}g\,\tilde{M}_{LJ}Q_L^{(\alpha\beta)}\,\overline{\varphi}_K-\mathrm{i}g\,\varphi_J\,\overline{Q}_L^{(\alpha\beta)}\tilde{M}_{LK}\right)\\
&\quad\cdot\left(\frac{\partial u_{KN}}{\partial x^\beta}a_{NI\alpha}\overline{u}_{IJ}+u_{KL}a_{LI\alpha}\frac{\partial\overline{u}_{IJ}}{\partial x^\beta}+\frac{1}{\mathrm{i}g}\frac{\partial u_{KI}}{\partial x^\alpha}\frac{\partial\overline{u}_{IJ}}{\partial x^\beta}+\frac{1}{\mathrm{i}g}\frac{\partial^2u_{KI}}{\partial x^\alpha\partial x^\beta}\overline{u}_{IJ}\right)\\
&\quad+\left(\tilde{M}_{LJ}Q_L^{(\alpha\beta)}\frac{\partial\overline{\varphi}_K}{\partial x^\beta}-\frac{\partial\varphi_J}{\partial x^\beta}\overline{Q}_L^{(\alpha\beta)}\tilde{M}_{LK}\right)\mathrm{i}g\,A_{KJ\alpha}\\
&\quad+\overline{Q}_L^{(\alpha\beta)}\tilde{M}_{LK}\left(\frac{\partial u_{KI}}{\partial x^\beta}M_{IJ}b_{J\alpha}+\frac{\partial^2\varphi_K}{\partial x^\alpha\partial x^\beta}\right)+\left(\overline{b}_{J\alpha}M_{IJ}\frac{\partial\overline{u}_{IK}}{\partial x^\beta}+\frac{\partial^2\overline{\varphi}_K}{\partial x^\alpha\partial x^\beta}\right)\tilde{M}_{LK}Q_L^{(\alpha\beta)}\\
&=\left(P_{JK}^{(\alpha\beta)}+\mathrm{i}g\,\tilde{M}_{LJ}Q_L^{(\alpha\beta)}\,\overline{\varphi}_K-\mathrm{i}g\,\varphi_J\,\overline{Q}_L^{(\alpha\beta)}\tilde{M}_{LK}\right)\left(\frac{\partial A_{KJ\alpha}}{\partial x^\beta}-u_{KL}\frac{\partial a_{LI\alpha}}{\partial x^\beta}\overline{u}_{IJ}\right)\\
&\quad+\overline{Q}_L^{(\alpha\beta)}\tilde{M}_{LK}\left(\frac{\partial u_{KI}}{\partial x^\beta}M_{IJ}b_{J\alpha}+\frac{\partial^2\varphi_K}{\partial x^\alpha\partial x^\beta}-\mathrm{i}g\,A_{KJ\alpha}\frac{\partial\varphi_J}{\partial x^\beta}\right)\\
&\quad+\left(\overline{b}_{J\alpha}M_{IJ}\frac{\partial\overline{u}_{IK}}{\partial x^\beta}+\frac{\partial^2\overline{\varphi}_K}{\partial x^\alpha\partial x^\beta}+\mathrm{i}g\,\frac{\partial\overline{\varphi}_J}{\partial x^\beta}A_{JK\alpha}\right)\tilde{M}_{LK}Q_L^{(\alpha\beta)}\\
&=\tfrac{1}{2}P_{JK}^{\alpha\beta}\left(\frac{\partial A_{KJ\alpha}}{\partial x^\beta}+\frac{\partial A_{KJ\beta}}{\partial x^\alpha}\right)+\tfrac{1}{2}\overline{Q}_K^{\alpha\beta}\left(\frac{\partial B_{K\alpha}}{\partial x^\beta}+\frac{\partial B_{K\beta}}{\partial x^\alpha}\right)+\tfrac{1}{2}\left(\frac{\partial\overline{B}_{K\alpha}}{\partial x^\beta}+\frac{\partial\overline{B}_{K\beta}}{\partial x^\alpha}\right)Q_K^{\alpha\beta}\\
&\quad-\tfrac{1}{2}p_{JK}^{\alpha\beta}\left(\frac{\partial a_{KJ\alpha}}{\partial x^\beta}+\frac{\partial a_{KJ\beta}}{\partial x^\alpha}\right)-\tfrac{1}{2}\overline{q}_K^{\alpha\beta}\left(\frac{\partial b_{K\alpha}}{\partial x^\beta}+\frac{\partial b_{K\beta}}{\partial x^\alpha}\right)-\tfrac{1}{2}\left(\frac{\partial\overline{b}_{K\alpha}}{\partial x^\beta}+\frac{\partial\overline{b}_{K\beta}}{\partial x^\alpha}\right)q_K^{\alpha\beta}.
\end{aligned}
\tag{29}
$$

In summary, by inserting the transformation rules into Eq. (26), the divergence of the explicitly x^μ-dependent terms of F_2^μ — and hence the difference of transformed and original Hamiltonians — can be expressed completely in terms of the canonical variables as

$$
\begin{aligned}
\left.\frac{\partial F_2^\alpha}{\partial x^\alpha}\right|_{\mathrm{expl}}&=\mathrm{i}g\left(\overline{\Pi}_K^\alpha\Phi_J-\overline{\Phi}_K\Pi_J^\alpha\right)A_{KJ\alpha}+\overline{\Pi}_K^\alpha M_{KJ}B_{J\alpha}+\overline{B}_{K\alpha}M_{JK}\Pi_J^\alpha\\
&\quad-\mathrm{i}g\left(\overline{\pi}_K^\alpha\phi_J-\overline{\phi}_K\pi_J^\alpha\right)a_{KJ\alpha}-\left(\overline{\pi}_K^\alpha M_{KJ}b_{J\alpha}+\overline{b}_{K\alpha}M_{JK}\pi_J^\alpha\right)\\
&\quad-\tfrac{1}{2}\mathrm{i}g\,P_{JK}^{\alpha\beta}\left(A_{KI\alpha}A_{IJ\beta}-A_{KI\beta}A_{IJ\alpha}\right)+\tfrac{1}{2}\mathrm{i}g\,p_{JK}^{\alpha\beta}\left(a_{KI\alpha}a_{IJ\beta}-a_{KI\beta}a_{IJ\alpha}\right)\\
&\quad+\tfrac{1}{2}\mathrm{i}g\left(\overline{B}_{J\beta}M_{KJ}A_{KI\alpha}\tilde{M}_{IL}-\overline{B}_{J\alpha}M_{KJ}A_{KI\beta}\tilde{M}_{LI}\right)Q_L^{\alpha\beta}\\
&\quad-\tfrac{1}{2}\mathrm{i}g\,\overline{Q}_L^{\alpha\beta}\left(\tilde{M}_{LI}A_{IK\alpha}M_{KJ}B_{J\beta}-\tilde{M}_{LI}A_{IK\beta}M_{KJ}B_{J\alpha}\right)\\
&\quad-\tfrac{1}{2}\mathrm{i}g\left(\overline{b}_{J\beta}M_{KJ}a_{KI\alpha}\tilde{M}_{IL}-\overline{b}_{J\alpha}M_{KJ}a_{KI\beta}\tilde{M}_{LI}\right)q_L^{\alpha\beta}
\end{aligned}
$$

$$+\tfrac{1}{2}\mathrm{i}g\,\overline{q}_L^{\alpha\beta}\left(\tilde{M}_{LI}a_{IK\alpha}M_{KJ}b_{J\beta}-\tilde{M}_{LI}a_{IK\beta}M_{KJ}b_{J\alpha}\right)$$
$$+\tfrac{1}{2}P_{JK}^{\alpha\beta}\left(\frac{\partial A_{KJ\alpha}}{\partial x^\beta}+\frac{\partial A_{KJ\beta}}{\partial x^\alpha}\right)+\tfrac{1}{2}\overline{Q}_K^{\alpha\beta}\left(\frac{\partial B_{K\alpha}}{\partial x^\beta}+\frac{\partial B_{K\beta}}{\partial x^\alpha}\right)+\tfrac{1}{2}\left(\frac{\partial \overline{B}_{K\alpha}}{\partial x^\beta}+\frac{\partial \overline{B}_{K\beta}}{\partial x^\alpha}\right)Q_K^{\alpha\beta}$$
$$-\tfrac{1}{2}p_{JK}^{\alpha\beta}\left(\frac{\partial a_{KJ\alpha}}{\partial x^\beta}+\frac{\partial a_{KJ\beta}}{\partial x^\alpha}\right)-\tfrac{1}{2}\overline{q}_K^{\alpha\beta}\left(\frac{\partial b_{K\alpha}}{\partial x^\beta}+\frac{\partial b_{K\beta}}{\partial x^\alpha}\right)-\tfrac{1}{2}\left(\frac{\partial \overline{b}_{K\alpha}}{\partial x^\beta}+\frac{\partial \overline{b}_{K\beta}}{\partial x^\alpha}\right)q_K^{\alpha\beta}.$$

We observe that *all* u_{IJ}-dependencies of Eq. (26) were expressed *symmetrically* in terms of both the original and the transformed complex base fields ϕ_J, Φ_J and 4-vector gauge fields $\boldsymbol{a}_{JK}, \boldsymbol{A}_{JK}$,$\boldsymbol{b}_J, \boldsymbol{B}_J$, in conjunction with their respective canonical momenta. Consequently, an amended Hamiltonian $\mathcal{H}_2$ of the form

$$\begin{aligned}\mathcal{H}_2=\mathcal{H}(\boldsymbol{\pi},\boldsymbol{\phi},x)&+\mathrm{i}g\left(\overline{\pi}_K^\alpha\phi_J-\overline{\phi}_K\pi_J^\alpha\right)a_{KJ\alpha}+\overline{\pi}_K^\alpha M_{KJ}b_{J\alpha}+\overline{b}_{K\alpha}M_{JK}\pi_J^\alpha\\
&-\tfrac{1}{2}\mathrm{i}g\,p_{JK}^{\alpha\beta}\left(a_{KI\alpha}\,a_{IJ\beta}-a_{KI\beta}\,a_{IJ\alpha}\right)+\tfrac{1}{2}p_{JK}^{\alpha\beta}\left(\frac{\partial a_{KJ\alpha}}{\partial x^\beta}+\frac{\partial a_{KJ\beta}}{\partial x^\alpha}\right)\\
&+\tfrac{1}{2}\mathrm{i}g\left(\overline{b}_{J\beta}M_{KJ}a_{KI\alpha}-\overline{b}_{J\alpha}M_{KJ}a_{KI\beta}\right)\tilde{M}_{LI}q_L^{\alpha\beta}\\
&-\tfrac{1}{2}\mathrm{i}g\,\overline{q}_L^{\alpha\beta}\tilde{M}_{LI}\left(a_{IK\alpha}M_{KJ}b_{J\beta}-a_{IK\beta}M_{KJ}b_{J\alpha}\right)\\
&+\tfrac{1}{2}\overline{q}_K^{\alpha\beta}\left(\frac{\partial b_{K\alpha}}{\partial x^\beta}+\frac{\partial b_{K\beta}}{\partial x^\alpha}\right)+\tfrac{1}{2}\left(\frac{\partial \overline{b}_{K\alpha}}{\partial x^\beta}+\frac{\partial \overline{b}_{K\beta}}{\partial x^\alpha}\right)q_K^{\alpha\beta}\end{aligned}\tag{30}$$

is then transformed according to the general rule (11)

$$\mathcal{H}_2'=\mathcal{H}_2+\left.\frac{\partial F_2^\alpha}{\partial x^\alpha}\right|_{\mathrm{expl}}$$

into the new Hamiltonian

$$\begin{aligned}\mathcal{H}_2'=\mathcal{H}(\boldsymbol{\Pi},\boldsymbol{\Phi},x)&+\mathrm{i}g\left(\overline{\Pi}_K^\alpha\Phi_J-\overline{\Phi}_K\Pi_J^\alpha\right)A_{KJ\alpha}+\overline{\Pi}_K^\alpha M_{KJ}B_{J\alpha}+\overline{B}_{K\alpha}M_{JK}\Pi_J^\alpha\\
&-\tfrac{1}{2}\mathrm{i}g\,P_{JK}^{\alpha\beta}\left(A_{KI\alpha}\,A_{IJ\beta}-A_{KI\beta}\,A_{IJ\alpha}\right)+\tfrac{1}{2}P_{JK}^{\alpha\beta}\left(\frac{\partial A_{KJ\alpha}}{\partial x^\beta}+\frac{\partial A_{KJ\beta}}{\partial x^\alpha}\right)\\
&+\tfrac{1}{2}\mathrm{i}g\left(\overline{B}_{J\beta}M_{KJ}A_{KI\alpha}-\overline{B}_{J\alpha}M_{KJ}A_{KI\beta}\right)\tilde{M}_{LI}Q_L^{\alpha\beta}\\
&-\tfrac{1}{2}\mathrm{i}g\,\overline{Q}_L^{\alpha\beta}\tilde{M}_{LI}\left(A_{IK\alpha}M_{KJ}B_{J\beta}-A_{IK\beta}M_{KJ}B_{J\alpha}\right)\\
&+\tfrac{1}{2}\overline{Q}_K^{\alpha\beta}\left(\frac{\partial B_{K\alpha}}{\partial x^\beta}+\frac{\partial B_{K\beta}}{\partial x^\alpha}\right)+\tfrac{1}{2}\left(\frac{\partial \overline{B}_{K\alpha}}{\partial x^\beta}+\frac{\partial \overline{B}_{K\beta}}{\partial x^\alpha}\right)Q_K^{\alpha\beta}.\end{aligned}\tag{31}$$

The entire transformation is thus *form-conserving* provided that the original Hamiltonian $\mathcal{H}(\boldsymbol{\pi},\boldsymbol{\phi},x)$ is also form-invariant if expressed in terms of the new fields, $\mathcal{H}(\boldsymbol{\Pi},\boldsymbol{\Phi},x)=\mathcal{H}(\boldsymbol{\pi},\boldsymbol{\phi},x)$, according to the transformation rules (14). In other words, $\mathcal{H}(\boldsymbol{\pi},\boldsymbol{\phi},x)$ must be form-invariant under the corresponding *global* gauge transformation.

As a common feature of all gauge transformation theories, we must ensure that the transformation rules for the gauge fields and their conjugates are consistent with the *field equations* for the gauge fields that follow from final form-invariant amended Hamiltonians, $\mathcal{H}_3=\mathcal{H}_2+\mathcal{H}_{\mathrm{dyn}}$ and $\mathcal{H}_3'=\mathcal{H}_2'+\mathcal{H}_{\mathrm{dyn}}'$. In other words, $\mathcal{H}_{\mathrm{dyn}}$ and the form-alike $\mathcal{H}_{\mathrm{dyn}}'$ must be chosen in a way that the transformation properties of the canonical equations for the gauge fields emerging from $\mathcal{H}_3$ and $\mathcal{H}_3'$ are compatible with the canonical transformation rules (18). These requirements *uniquely determine* the form of both $\mathcal{H}_{\mathrm{dyn}}$ and $\mathcal{H}_{\mathrm{dyn}}'$. Thus, the

Hamiltonians (30) and (31) must be further amended by terms $\mathcal{H}_{\mathrm{dyn}}$ and $\mathcal{H}'_{\mathrm{dyn}}$ that describe the dynamics of the free 4-vector gauge fields, $\boldsymbol{a}_{KJ}, \boldsymbol{b}_J$ and $\boldsymbol{A}_{KJ}, \boldsymbol{B}_J$, respectively. Of course, $\mathcal{H}_{\mathrm{dyn}}$ must be form-invariant as well if expressed in the transformed dynamical variables in order to ensure the overall form-invariance of the final Hamiltonian. An expression that fulfils this requirement is obtained from Eqs. (21) and (22)

$$\begin{aligned}\mathcal{H}_{\mathrm{dyn}} = -\tfrac{1}{2}\overline{q}_J^{\alpha\beta}\, q_{J\alpha\beta} - \tfrac{1}{4}\left(p_{IJ}^{\alpha\beta} + \mathrm{i}g\,\tilde{M}_{LI} q_L^{\alpha\beta}\,\overline{\phi}_J - \mathrm{i}g\,\phi_I\,\overline{q}_L^{\alpha\beta}\tilde{M}_{LJ}\right)\\ \cdot\left(p_{JI\alpha\beta} + \mathrm{i}g\,\tilde{M}_{KJ} q_{K\alpha\beta}\,\overline{\phi}_I - \mathrm{i}g\,\phi_J\,\overline{q}_{K\alpha\beta}\tilde{M}_{KI}\right).\end{aligned} \tag{32}$$

The condition for the first term to be form-invariant is

$$\begin{aligned}\overline{q}_J^{\alpha\beta}\, q_{J\alpha\beta} &= \overline{Q}_L^{\alpha\beta}\,\tilde{M}_{LK}\, u_{KI}\;\underbrace{M_{IJ}\, M_{NJ}}_{\stackrel{!}{=}\delta_{IN}(\det M)^2}\;\overline{u}_{NR}\,\tilde{M}_{SR}\, Q_{S\alpha\beta}\\ &= (\det M)^2\,\overline{Q}_L^{\alpha\beta}\;\underbrace{\tilde{M}_{LK}\,\tilde{M}_{JK}}_{\stackrel{!}{=}\delta_{LJ}(\det M)^{-2}}\;Q_{J\alpha\beta}\\ &= \overline{Q}_J^{\alpha\beta}\, Q_{J\alpha\beta}\end{aligned}$$

The mass matrix M must thus be orthogonal

$$M\, M^T = 1\,(\det M)^2. \tag{33}$$

From $\mathcal{H}_3$ and, correspondingly, from $\mathcal{H}'_3$, we will work out the condition for the canonical field equations to be consistent with the canonical transformation rules (18) for the gauge fields and their conjugates (21).

With $\mathcal{H}_{\mathrm{dyn}}$ from Eq. (32), the total amended Hamiltonian $\mathcal{H}_3$ is now given by

$$\mathcal{H}_3 = \mathcal{H}_2 + \mathcal{H}_{\mathrm{dyn}} = \mathcal{H} + \mathcal{H}_{\mathrm{g}} \tag{34}$$

$$\begin{aligned}\mathcal{H}_{\mathrm{g}} = {}& \mathrm{i}g\left(\overline{\pi}_K^{\alpha}\phi_J - \overline{\phi}_K\pi_J^{\alpha}\right) a_{KJ\alpha} - \tfrac{1}{2}\mathrm{i}g\, p_{KJ}^{\alpha\beta}\left(a_{JI\alpha}\, a_{IK\beta} - a_{JI\beta}\, a_{IK\alpha}\right)\\ &+ \tfrac{1}{2}p_{KJ}^{\alpha\beta}\left(\frac{\partial a_{JK\alpha}}{\partial x^{\beta}} + \frac{\partial a_{JK\beta}}{\partial x^{\alpha}}\right) + \tfrac{1}{2}\overline{q}_J^{\alpha\beta}\left(\frac{\partial b_{J\alpha}}{\partial x^{\beta}} + \frac{\partial b_{J\beta}}{\partial x^{\alpha}}\right) + \tfrac{1}{2}\left(\frac{\partial \overline{b}_{J\alpha}}{\partial x^{\beta}} + \frac{\partial \overline{b}_{J\beta}}{\partial x^{\alpha}}\right) q_J^{\alpha\beta}\\ &+ \overline{\pi}_K^{\alpha} M_{KJ} b_{J\alpha} + \overline{b}_{K\alpha} M_{JK}\pi_J^{\alpha} + \tfrac{1}{2}\mathrm{i}g\left(\overline{b}_{J\beta} M_{KJ} a_{KI\alpha} - \overline{b}_{J\alpha} M_{KJ} a_{KI\beta}\right)\tilde{M}_{LI} q_L^{\alpha\beta}\\ &- \tfrac{1}{2}\mathrm{i}g\,\overline{q}_L^{\alpha\beta}\tilde{M}_{LI}\left(a_{IK\alpha} M_{KJ} b_{J\beta} - a_{IK\beta} M_{KJ} b_{J\alpha}\right) - \tfrac{1}{2}\overline{q}_J^{\alpha\beta}\, q_{J\alpha\beta}\\ &- \tfrac{1}{4}\left(p_{IJ}^{\alpha\beta} + \mathrm{i}g\,\tilde{M}_{LI} q_L^{\alpha\beta}\,\overline{\phi}_J - \mathrm{i}g\,\phi_I\,\overline{q}_L^{\alpha\beta}\tilde{M}_{LJ}\right)\\ &\qquad\cdot\left(p_{JI\alpha\beta} + \mathrm{i}g\,\tilde{M}_{KJ} q_{K\alpha\beta}\,\overline{\phi}_I - \mathrm{i}g\,\phi_J\,\overline{q}_{K\alpha\beta}\tilde{M}_{KI}\right).\end{aligned}$$

We reiterate that the system Hamiltonian $\mathcal{H}$ must be invariant under the corresponding *global* gauge transformation, hence a transformation of the form of Eq. (14) with the u_{IK} *not* depending on x.

In the Hamiltonian description, the partial derivatives of the fields in (34) do *not* constitute canonical variables and must hence be regarded as x^{μ}-dependent coefficients when setting up the canonical field equations. The relation of the canonical momenta $p_{NM}^{\mu\nu}$ to the derivatives of the fields, $\partial a_{MN\mu}/\partial x^{\nu}$, is generally provided by the first canonical field

equation (5). This means for the particular Hamiltonian (34)

$$\begin{aligned}\frac{\partial a_{MN\mu}}{\partial x^{\nu}} &= \frac{\partial \mathcal{H}_{\mathrm{g}}}{\partial p_{NM}^{\mu\nu}} \\ &= -\tfrac{1}{2}\mathrm{i}g\left(a_{MI\mu}\,a_{IN\nu} - a_{MI\nu}\,a_{IN\mu}\right) + \tfrac{1}{2}\left(\frac{\partial a_{MN\mu}}{\partial x^{\nu}} + \frac{\partial a_{MN\nu}}{\partial x^{\mu}}\right) \\ &\quad - \tfrac{1}{2}p_{MN\mu\nu} - \tfrac{1}{2}\mathrm{i}g\left(\tilde{M}_{IM}q_{I\mu\nu}\,\overline{\phi}_N - \phi_M\,\overline{q}_{I\mu\nu}\tilde{M}_{IN}\right),\end{aligned}$$

hence

$$\begin{aligned}p_{KJ\mu\nu} &= \frac{\partial a_{KJ\nu}}{\partial x^{\mu}} - \frac{\partial a_{KJ\mu}}{\partial x^{\nu}} \\ &\quad + \mathrm{i}g\left(a_{KI\nu}\,a_{IJ\mu} - a_{KI\mu}\,a_{IJ\nu} - \tilde{M}_{IK}q_{I\mu\nu}\,\overline{\phi}_J + \phi_K\,\overline{q}_{I\mu\nu}\tilde{M}_{IJ}\right).\end{aligned} \tag{35}$$

Rewriting Eq. (35) in the form

$$\begin{aligned}p_{KJ\mu\nu} + \mathrm{i}g\tilde{M}_{IK}q_{I\mu\nu}\overline{\phi}_J - \mathrm{i}g\phi_K\overline{q}_{I\mu\nu}\tilde{M}_{IJ} &= \frac{\partial a_{KJ\nu}}{\partial x^{\mu}} - \frac{\partial a_{KJ\mu}}{\partial x^{\nu}} + \mathrm{i}g\left(a_{KI\nu}a_{IJ\mu} - a_{KI\mu}a_{IJ\nu}\right) \\ &= f_{KJ\mu\nu},\end{aligned}$$

we realise that the left-hand side transforms homogeneously according to Eq. (22). From Eq. (25), we already know that the same rule applies for the $\boldsymbol{f}_{\mu\nu}$. The canonical equation (35) is thus generally consistent with the canonical transformation rules.

The corresponding reasoning applies for the canonical momenta $q_{J\mu\nu}$ and $\overline{q}_{J\mu\nu}$

$$\begin{aligned}\frac{\partial b_{N\mu}}{\partial x^{\nu}} &= \frac{\partial \mathcal{H}_{\mathrm{g}}}{\partial \overline{q}_{N}^{\mu\nu}} = -\tfrac{1}{2}q_{N\mu\nu} - \tfrac{1}{2}\mathrm{i}g\,\tilde{M}_{NI}\left(a_{IK\mu}M_{KJ}\,b_{J\nu} - a_{IK\nu}M_{KJ}\,b_{J\mu}\right) \\ &\quad + \tfrac{1}{2}\left(\frac{\partial b_{N\mu}}{\partial x^{\nu}} + \frac{\partial b_{N\nu}}{\partial x^{\mu}}\right) + \tfrac{1}{2}\mathrm{i}g\,\tilde{M}_{NI}\left(p_{IJ\mu\nu} + \mathrm{i}g\,\tilde{M}_{KI}q_{K\mu\nu}\,\overline{\phi}_J - \mathrm{i}g\,\phi_I\,\overline{q}_{K\mu\nu}\tilde{M}_{KJ}\right)\phi_J \\ \frac{\partial \overline{b}_{N\mu}}{\partial x^{\nu}} &= \frac{\partial \mathcal{H}_{\mathrm{g}}}{\partial q_{N}^{\mu\nu}} = -\tfrac{1}{2}\overline{q}_{N\mu\nu} + \tfrac{1}{2}\mathrm{i}g\left(\overline{b}_{J\nu}M_{KJ}\,a_{KI\mu} - \overline{b}_{J\mu}M_{KJ}\,a_{KI\nu}\right)\tilde{M}_{NI} \\ &\quad + \tfrac{1}{2}\left(\frac{\partial \overline{b}_{N\mu}}{\partial x^{\nu}} + \frac{\partial \overline{b}_{N\nu}}{\partial x^{\mu}}\right) - \tfrac{1}{2}\mathrm{i}g\,\overline{\phi}_J\left(p_{JI\mu\nu} + \mathrm{i}g\,\tilde{M}_{KJ}q_{K\mu\nu}\,\overline{\phi}_I - \mathrm{i}g\,\phi_J\,\overline{q}_{K\mu\nu}\tilde{M}_{KI}\right)\tilde{M}_{NI},\end{aligned}$$

hence with the canonical equation (35)

$$\begin{aligned}q_{J\mu\nu} &= \frac{\partial b_{J\nu}}{\partial x^{\mu}} - \frac{\partial b_{J\mu}}{\partial x^{\nu}} + \mathrm{i}g\,\tilde{M}_{JI}\left(a_{IK\nu}M_{KL}\,b_{L\mu} - a_{IK\mu}M_{KL}\,b_{L\nu}\right) \\ &\quad + \mathrm{i}g\,\tilde{M}_{JI}\left(\frac{\partial a_{IK\nu}}{\partial x^{\mu}} - \frac{\partial a_{IK\mu}}{\partial x^{\nu}} + \mathrm{i}g\left(a_{IL\nu}\,a_{LK\mu} - a_{IL\mu}\,a_{LK\nu}\right)\right)\phi_K \\ \overline{q}_{J\mu\nu} &= \frac{\partial \overline{b}_{J\nu}}{\partial x^{\mu}} - \frac{\partial \overline{b}_{J\mu}}{\partial x^{\nu}} - \mathrm{i}g\left(\overline{b}_{L\mu}M_{KL}\,a_{KI\nu} - \overline{b}_{L\nu}M_{KL}\,a_{KI\mu}\right)\tilde{M}_{JI} \\ &\quad - \mathrm{i}g\,\overline{\phi}_K\left(\frac{\partial a_{KI\nu}}{\partial x^{\mu}} - \frac{\partial a_{KI\mu}}{\partial x^{\nu}} + \mathrm{i}g\left(a_{KL\nu}\,a_{LI\mu} - a_{KL\mu}\,a_{LI\nu}\right)\right)\tilde{M}_{JI}.\end{aligned} \tag{36}$$

In order to check whether these canonical equations — which are complex conjugate to each other – are also compatible with the canonical transformation rules, we rewrite the first one concisely in matrix notation for the transformed fields

$$M\boldsymbol{Q}_{\mu\nu} = \frac{\partial M\boldsymbol{B}_{\nu}}{\partial x^{\mu}} - \frac{\partial M\boldsymbol{B}_{\mu}}{\partial x^{\nu}} + \mathrm{i}g\left(\boldsymbol{A}_{\nu}M\,\boldsymbol{B}_{\mu} - \boldsymbol{A}_{\mu}M\,\boldsymbol{B}_{\nu}\right)$$

$$+\,\mathrm{i}g\left(\frac{\partial\boldsymbol{A}_\nu}{\partial x^\mu}-\frac{\partial\boldsymbol{A}_\mu}{\partial x^\nu}+\mathrm{i}g\left(\boldsymbol{A}_\nu\boldsymbol{A}_\mu-\boldsymbol{A}_\mu\boldsymbol{A}_\nu\right)\right)\boldsymbol{\Phi}.$$

Applying now the transformation rules for the gauge fields $\boldsymbol{A}_\nu,\boldsymbol{B}_\mu$ from Eqs. (19), and for the base fields $\boldsymbol{\Phi}$ from Eqs. (12), we find

$$\begin{aligned}M\boldsymbol{Q}_{\mu\nu}&=U\left[\frac{\partial M\boldsymbol{b}_\nu}{\partial x^\mu}-\frac{\partial M\boldsymbol{b}_\mu}{\partial x^\nu}+\mathrm{i}g\left(\boldsymbol{a}_\nu M\,\boldsymbol{b}_\mu-\boldsymbol{a}_\mu M\,\boldsymbol{b}_\nu\right)\right.\\&\qquad\left.+\,\mathrm{i}g\left(\frac{\partial\boldsymbol{a}_\nu}{\partial x^\mu}-\frac{\partial\boldsymbol{a}_\mu}{\partial x^\nu}+\mathrm{i}g\left(\boldsymbol{a}_\nu\boldsymbol{a}_\mu-\boldsymbol{a}_\mu\boldsymbol{a}_\nu\right)\right)\boldsymbol{\phi}\right]\\&=UM\,\boldsymbol{q}_{\mu\nu}.\end{aligned}$$

The canonical equations (36) are thus compatible with the canonical transformation rules (25) provided that

$$\tilde{M}^T=\frac{M}{\left(\det M\right)^2}.$$

Thus, the mass matrix M must be *orthogonal*. This restriction was already encountered with Eq. (33).

We observe that both $p_{KJ\mu\nu}$ and $q_{J\mu\nu},\overline{q}_{J\mu\nu}$ occur to be skew-symmetric in the indices μ,ν. Here, this feature emerges from the canonical formalism and does not have to be postulated. Consequently, all products with the momenta in the Hamiltonian (34) that are *symmetric* in μ,ν must vanish. As these terms only contribute to the first canonical equations, we may omit them from $\mathcal{H}_\mathrm{g}$ if we simultaneously *define* $p_{JK\mu\nu}$ and $q_{J\mu\nu}$ to be skew-symmetric in μ,ν. With regard to the ensuing canonical equations, the gauge Hamiltonian $\mathcal{H}_\mathrm{g}$ from Eq. (34) is then equivalent to

$$\begin{aligned}\mathcal{H}_\mathrm{g}&=\mathrm{i}g\left(\overline{\pi}_K^{\beta}\phi_J-\overline{\phi}_K\pi_J^{\beta}\right)a_{KJ\beta}-\mathrm{i}g\,p_{JI}^{\alpha\beta}a_{IK\alpha}\,a_{KJ\beta}-\tfrac{1}{2}\,\overline{q}_J^{\,\alpha\beta}\,q_{J\alpha\beta}\\&\quad+\left(\overline{\pi}_K^{\beta}-\mathrm{i}g\,\overline{q}_L^{\,\alpha\beta}\tilde{M}_{LI}a_{IK\alpha}\right)M_{KJ}b_{J\beta}+\overline{b}_{K\beta}M_{JK}\left(\pi_J^{\beta}+\mathrm{i}g\,a_{JI\alpha}\tilde{M}_{LI}q_L^{\alpha\beta}\right)\\&\quad-\tfrac{1}{4}\left(p_{IJ}^{\alpha\beta}+\mathrm{i}g\,\tilde{M}_{LI}q_L^{\alpha\beta}\,\overline{\phi}_J-\mathrm{i}g\,\phi_I\,\overline{q}_L^{\,\alpha\beta}\tilde{M}_{LJ}\right)\\&\quad\;\cdot\left(p_{JI\alpha\beta}+\mathrm{i}g\,\tilde{M}_{KJ}q_{K\alpha\beta}\,\overline{\phi}_I-\mathrm{i}g\,\phi_J\,\overline{q}_{K\alpha\beta}\tilde{M}_{KI}\right)\\&\quad p_{JK}^{\mu\nu}\stackrel{!}{=}-p_{JK}^{\nu\mu},\qquad q_J^{\mu\nu}\stackrel{!}{=}-q_J^{\nu\mu}.\end{aligned}\tag{37}$$

Setting the mass matrix M to zero, $\mathcal{H}_\mathrm{g}$ reduces to the gauge Hamiltonian of the homogeneous $\mathrm{U}(N)$ gauge theory (Struckmeier and Reichau 2012). The other terms describe the dynamics of the 4-vector gauge fields $\boldsymbol{b}_J$. From the locally gauge-invariant Hamiltonian (34), the canonical equations for the base fields $\phi_I,\overline{\phi}_I$ are given by

$$\begin{aligned}\left.\frac{\partial\phi_I}{\partial x^\mu}\right|_{\mathcal{H}_3}&=\frac{\partial\mathcal{H}_3}{\partial\overline{\pi}_I^{\mu}}=\frac{\partial\mathcal{H}}{\partial\overline{\pi}_I^{\mu}}+\mathrm{i}g\,a_{IJ\mu}\phi_J+M_{IJ}b_{J\mu}\\\left.\frac{\partial\overline{\phi}_I}{\partial x^\mu}\right|_{\mathcal{H}_3}&=\frac{\partial\mathcal{H}_3}{\partial\pi_I^{\mu}}=\frac{\partial\mathcal{H}}{\partial\pi_I^{\mu}}-\mathrm{i}g\,\overline{\phi}_Ja_{JI\mu}+\overline{b}_{J\mu}M_{IJ}.\end{aligned}\tag{38}$$

These equations represent the generalised "minimum coupling rules" for our particular case of a system of two sets of gauge fields, $\boldsymbol{a}_{JK}$ and $\boldsymbol{b}_J$.

The canonical field equation from the $\boldsymbol{b}_J, \overline{\boldsymbol{b}}_J$ dependencies of $\mathcal{H}_{\mathrm{g}}$ follow as

$$\frac{\partial q_K^{\mu\alpha}}{\partial x^\alpha} = -\frac{\partial \mathcal{H}_{\mathrm{g}}}{\partial \overline{b}_{K\mu}} = -M_{JK}\left(\pi_J^\mu + \mathrm{i}g\, a_{JI\alpha} \tilde{M}_{LI} q_L^{\alpha\mu}\right)$$
$$\frac{\partial \overline{q}_J^{\mu\alpha}}{\partial x^\alpha} = -\frac{\partial \mathcal{H}_{\mathrm{g}}}{\partial b_{J\mu}} = \left(-\overline{\pi}_K^\mu + \mathrm{i}g\, \overline{q}_L^{\alpha\mu} \tilde{M}_{LI} a_{IK\alpha}\right) M_{KJ}.$$

Inserting $\pi_J^\alpha, \overline{\pi}_J^\alpha$ as obtained from Eqs. (38) for a particular system Hamiltonian $\mathcal{H}$, terms proportional to b_I^α and $\overline{b}_I^\alpha$ emerge with no other dynamical variables involved. Such terms describe the masses of particles that are associated with the gauge fields $\boldsymbol{b}_I$.

4.3. Gauge-invariant Lagrangian

As the system Hamiltonian $\mathcal{H}$ does not depend on the gauge fields $\boldsymbol{a}_{KJ}$ and $\boldsymbol{b}_J$, the gauge Lagrangian $\mathcal{L}_{\mathrm{g}}$ that is equivalent to the gauge Hamiltonian $\mathcal{H}_{\mathrm{g}}$ from Eq. (34) is derived by means of the Legendre transformation

$$\mathcal{L}_{\mathrm{g}} = p_{JK}^{\alpha\beta} \frac{\partial a_{KJ\alpha}}{\partial x^\beta} + \overline{q}_J^{\alpha\beta} \frac{\partial b_{J\alpha}}{\partial x^\beta} + \frac{\partial \overline{b}_{J\alpha}}{\partial x^\beta} q_J^{\alpha\beta} - \mathcal{H}_{\mathrm{g}},$$

with $p_{JK}^{\mu\nu}$ from Eq. (35) and $q_J^{\mu\nu}, \overline{q}_J^{\mu\nu}$ from Eqs. (36). We thus have

$$\begin{aligned}
p_{JK}^{\alpha\beta} \frac{\partial a_{KJ\alpha}}{\partial x^\beta} &= \tfrac{1}{2} p_{JK}^{\alpha\beta} \left(\frac{\partial a_{KJ\alpha}}{\partial x^\beta} - \frac{\partial a_{KJ\beta}}{\partial x^\alpha}\right) + \tfrac{1}{2} p_{JK}^{\alpha\beta} \left(\frac{\partial a_{KJ\alpha}}{\partial x^\beta} + \frac{\partial a_{KJ\beta}}{\partial x^\alpha}\right) \\
&= -\tfrac{1}{2} p_{JK}^{\alpha\beta}\, p_{KJ\alpha\beta} + \tfrac{1}{2} p_{JK}^{\alpha\beta} \left(\frac{\partial a_{KJ\alpha}}{\partial x^\beta} + \frac{\partial a_{KJ\beta}}{\partial x^\alpha}\right) \\
&\quad - \tfrac{1}{2}\mathrm{i}g\, p_{JK}^{\alpha\beta} \left(a_{KI\alpha}\, a_{IJ\beta} - a_{KI\beta}\, a_{IJ\alpha} - \tilde{M}_{IK} q_{I\beta\alpha}\, \overline{\phi}_J + \phi_K\, \overline{q}_{I\beta\alpha} \tilde{M}_{IJ}\right),
\end{aligned}$$

and, similarly

$$\begin{aligned}
\overline{q}_J^{\alpha\beta} \frac{\partial b_{J\alpha}}{\partial x^\beta} &= -\tfrac{1}{2}\, \overline{q}_J^{\alpha\beta}\, q_{J\alpha\beta} - \tfrac{1}{2}\mathrm{i}g\, \overline{q}_J^{\alpha\beta} \tilde{M}_{JI} \left(a_{IK\alpha} M_{KL}\, b_{L\beta} - a_{IK\beta} M_{KL}\, b_{L\alpha}\right) \\
&\quad + \tfrac{1}{2}\mathrm{i}g\, \overline{q}_J^{\alpha\beta} \tilde{M}_{JI} \left(p_{IL\alpha\beta} + \mathrm{i}g\, \tilde{M}_{KI} q_{K\alpha\beta}\, \overline{\phi}_L - \mathrm{i}g\, \phi_I\, \overline{q}_{K\alpha\beta} \tilde{M}_{KL}\right) \phi_L \\
&\quad + \tfrac{1}{2} \overline{q}_J^{\alpha\beta} \left(\frac{\partial b_{J\alpha}}{\partial x^\beta} + \frac{\partial b_{J\beta}}{\partial x^\alpha}\right) \\
\frac{\partial \overline{b}_{J\alpha}}{\partial x^\beta} q_J^{\alpha\beta} &= -\tfrac{1}{2}\, \overline{q}_J^{\alpha\beta}\, q_{J\alpha\beta} + \tfrac{1}{2}\mathrm{i}g\, \left(\overline{b}_{L\beta} M_{KL}\, a_{KI\alpha} - \overline{b}_{L\alpha} M_{KL}\, a_{KI\beta}\right) \tilde{M}_{JI} q_J^{\alpha\beta} \\
&\quad - \tfrac{1}{2}\mathrm{i}g\, \overline{\phi}_I \left(p_{IL\alpha\beta} + \mathrm{i}g\, \tilde{M}_{KI} q_{K\alpha\beta}\, \overline{\phi}_L - \mathrm{i}g\, \phi_I\, \overline{q}_{K\alpha\beta} \tilde{M}_{KL}\right) \tilde{M}_{JL} q_J^{\alpha\beta} \\
&\quad + \tfrac{1}{2} \left(\frac{\partial \overline{b}_{J\alpha}}{\partial x^\beta} + \frac{\partial \overline{b}_{J\beta}}{\partial x^\alpha}\right) q_J^{\alpha\beta}.
\end{aligned}$$

With the gauge Hamiltonian $\mathcal{H}_{\mathrm{g}}$ from Eq. (34), the gauge Lagrangian $\mathcal{L}_{\mathrm{g}}$ is then

$$\begin{aligned}
\mathcal{L}_{\mathrm{g}} = &-\tfrac{1}{2}\, \overline{q}_J^{\alpha\beta} q_{J\alpha\beta} - \overline{\pi}_K^\alpha \left(\mathrm{i}g\, a_{KJ\alpha} \phi_J + M_{KJ} b_{J\alpha}\right) + \left(\mathrm{i}g\, \overline{\phi}_K a_{KJ\alpha} - \overline{b}_{K\alpha} M_{JK}\right) \pi_J^\alpha \\
&- \tfrac{1}{4} \left(p_{IJ}^{\alpha\beta} + \mathrm{i}g\, \tilde{M}_{LI} q_L^{\alpha\beta}\, \overline{\phi}_J - \mathrm{i}g\, \phi_I\, \overline{q}_L^{\alpha\beta} \tilde{M}_{LJ}\right) \\
&\cdot \left(p_{JI\alpha\beta} + \mathrm{i}g\, \tilde{M}_{KJ} q_{K\alpha\beta}\, \overline{\phi}_I - \mathrm{i}g\, \phi_J\, \overline{q}_{K\alpha\beta} \tilde{M}_{KI}\right)
\end{aligned}$$

According to Eq. (23) and the relation for the canonical momenta $p_{JI\alpha\beta}$ from Eq. (35), the last product can be rewritten as $-\frac{1}{4}f_{IJ}^{\alpha\beta}\,f_{JI\alpha\beta}$, thus

$$\mathcal{L}_{\mathrm{g}} = -\tfrac{1}{4}f_{IJ}^{\alpha\beta}\,f_{JI\alpha\beta} - \tfrac{1}{2}\,\overline{q}_{J}^{\alpha\beta}q_{J\alpha\beta} - \overline{\pi}_{K}^{\alpha}\left(\mathrm{i}g\,a_{KJ\alpha}\phi_J + M_{KJ}b_{J\alpha}\right) + \left(\mathrm{i}g\,\overline{\phi}_K a_{KJ\alpha} - \overline{b}_{K\alpha}M_{JK}\right)\pi_J^{\alpha}.$$

With regard to canonical variables $\overline{\boldsymbol{\pi}}_K, \boldsymbol{\pi}_K$, $\mathcal{L}_{\mathrm{g}}$ is still a Hamiltonian. The final total gauge-invariant Lagrangian $\mathcal{L}_3$ for the given system Hamiltonian $\mathcal{H}$ then emerges from the Legendre transformation

$$\begin{aligned}\mathcal{L}_3 &= \mathcal{L}_{\mathrm{g}} + \overline{\pi}_J^{\alpha}\frac{\partial\phi_J}{\partial x^{\alpha}} + \frac{\partial\overline{\phi}_J}{\partial x^{\alpha}}\pi_J^{\alpha} - \mathcal{H}(\overline{\phi}_I, \phi_I, \overline{\boldsymbol{\pi}}_I, \boldsymbol{\pi}_I, x)\\ &= \overline{\pi}_J^{\alpha}\left(\frac{\partial\phi_J}{\partial x^{\alpha}} - \mathrm{i}g\,a_{JK\alpha}\phi_K - M_{JK}\,b_{K\alpha}\right) + \left(\frac{\partial\overline{\phi}_J}{\partial x^{\alpha}} + \mathrm{i}g\,\overline{\phi}_K a_{KJ\alpha} - \overline{b}_{K\alpha}M_{JK}\right)\pi_J^{\alpha}\\ &\quad - \tfrac{1}{4}f_{IJ}^{\alpha\beta}\,f_{JI\alpha\beta} - \tfrac{1}{2}\,\overline{q}_J^{\alpha\beta}q_{J\alpha\beta} - \mathcal{H}(\overline{\phi}_I, \phi_I, \overline{\boldsymbol{\pi}}_I, \boldsymbol{\pi}_I, x).\end{aligned} \tag{39}$$

As implied by the Lagrangian formalism, the dynamical variables are given by both the fields, ϕ_I, $\overline{\phi}_I$, $\boldsymbol{a}_{KJ}$, $\boldsymbol{b}_J$, and $\overline{\boldsymbol{b}}_J$, and their respective partial derivatives with respect to the independent variables, x^{μ}. Therefore, the momenta $\boldsymbol{q}_J$ and $\overline{\boldsymbol{q}}_J$ of the Hamiltonian description are no longer dynamical variables in $\mathcal{L}_{\mathrm{g}}$ but merely *abbreviations* for combinations of the Lagrangian dynamical variables, which are here given by Eqs. (36). The correlation of the momenta $\boldsymbol{\pi}_I, \overline{\boldsymbol{\pi}}_I$ of the base fields $\phi_I, \overline{\phi}_I$ to their derivatives are derived from the system Hamiltonian $\mathcal{H}$ via

$$\begin{aligned}\frac{\partial\phi_I}{\partial x^{\mu}} &= \frac{\partial\mathcal{H}}{\partial\overline{\pi}_I^{\mu}} + \mathrm{i}g\,a_{IJ\mu}\phi_J + M_{IJ}b_{J\mu}\\ \frac{\partial\overline{\phi}_I}{\partial x^{\mu}} &= \frac{\partial\mathcal{H}}{\partial\pi_I^{\mu}} - \mathrm{i}g\,\overline{\phi}_J\,a_{JI\mu} + \overline{b}_{J\mu}M_{IJ},\end{aligned} \tag{40}$$

which represents the "minimal coupling rule" for our particular system. Thus, for any *globally* gauge-invariant Hamiltonian $\mathcal{H}(\phi_I, \boldsymbol{\pi}_I, x)$, the amended Lagrangian (39) with Eqs. (40) describes in the Lagrangian formalism the associated physical system that is invariant under *local* gauge transformations.

4.4. Klein-Gordon system Hamiltonian

As an example, we consider the generalised Klein-Gordon Hamiltonian (Struckmeier and Reichau 2012) that describes an N-tuple of *massless* spin-0 fields

$$\mathcal{H}_{\mathrm{KG}} = \overline{\pi}_I^{\,\alpha}\,\pi_{I\alpha}.$$

This Hamiltonian is clearly invariant under the inhomogeneous global gauge transformation (14). The reason for defining a *massless* system Hamiltonian $\mathcal{H}$ is that a mass term of the form $\overline{\phi}_I M_{JI} M_{JK}\phi_K$ that is contained in the general Klein-Gordon Hamiltonian is *not invariant* under the inhomogeneous gauge transformation from Eq. (14). According to Eqs. (39) and (40), the corresponding locally gauge-invariant Lagrangian $\mathcal{L}_{3,\mathrm{KG}}$ is then

$$\boxed{\mathcal{L}_{3,\mathrm{KG}} = \overline{\pi}_I^{\,\alpha}\,\pi_{I\alpha} - \tfrac{1}{4}f_{JK}^{\alpha\beta}\,f_{KJ\alpha\beta} - \tfrac{1}{2}\overline{q}_J^{\,\alpha\beta}q_{J\alpha\beta},} \tag{41}$$

with

$$
\begin{aligned}
f_{KJ\mu\nu} &= \frac{\partial a_{KJ\nu}}{\partial x^\mu} - \frac{\partial a_{KJ\mu}}{\partial x^\nu} + \mathrm{i}g\left(a_{KI\nu}\,a_{IJ\mu} - a_{KI\mu}\,a_{IJ\nu}\right)\\
q_{J\mu\nu} &= \frac{\partial b_{J\nu}}{\partial x^\mu} - \frac{\partial b_{J\mu}}{\partial x^\nu} + \mathrm{i}g\,\tilde{M}_{JI}\left(a_{IK\nu}\,M_{KL}\,b_{L\mu} - a_{IK\mu}M_{KL}\,b_{L\nu} + f_{IK\mu\nu}\,\phi_K\right)\\
\overline{q}_{J\mu\nu} &= \frac{\partial \overline{b}_{J\nu}}{\partial x^\mu} - \frac{\partial \overline{b}_{J\mu}}{\partial x^\nu} - \mathrm{i}g\left(\overline{b}_{L\mu}\,M_{KL}\,a_{KI\nu} - \overline{b}_{L\nu}\,M_{KL}\,a_{KI\mu} + \overline{\phi}_K\,f_{KI\mu\nu}\right)\tilde{M}_{JI}\\
\pi_{I\mu} &= \frac{\partial \phi_I}{\partial x^\mu} - \mathrm{i}g\,a_{IJ\mu}\phi_J - M_{IJ}\,b_{J\mu}\\
\overline{\pi}_{I\mu} &= \frac{\partial \overline{\phi}_I}{\partial x^\mu} + \mathrm{i}g\,\overline{\phi}_J\,a_{JI\mu} - \overline{b}_{J\mu}\,M_{IJ}.
\end{aligned}
$$

In matrix notation, the gauge-invariant Lagrangian (41) thus writes

$$
\begin{aligned}
\mathcal{L}_{3,\mathrm{KG}} = &\left(\frac{\partial \overline{\boldsymbol{\phi}}}{\partial x_\alpha} + \mathrm{i}g\,\overline{\boldsymbol{\phi}}\,\boldsymbol{a}^\alpha - \overline{\boldsymbol{b}}^\alpha M^T\right)\left(\frac{\partial \boldsymbol{\phi}}{\partial x^\alpha} - \mathrm{i}g\,\boldsymbol{a}_\alpha\boldsymbol{\phi} - M\boldsymbol{b}_\alpha\right)\\
&- \mathrm{Tr}\left(\tfrac{1}{4}\boldsymbol{f}^{\alpha\beta}\boldsymbol{f}_{\alpha\beta}\right) - \tfrac{1}{2}\overline{\boldsymbol{q}}^{\alpha\beta}\boldsymbol{q}_{\alpha\beta}
\end{aligned}
$$

with

$$
\begin{aligned}
\boldsymbol{f}_{\mu\nu} &= \frac{\partial \boldsymbol{a}_\nu}{\partial x^\mu} - \frac{\partial \boldsymbol{a}_\mu}{\partial x^\nu} + \mathrm{i}g\left(\boldsymbol{a}_\nu\,\boldsymbol{a}_\mu - \boldsymbol{a}_\mu\,\boldsymbol{a}_\nu\right)\\
M\boldsymbol{q}_{\mu\nu} &= M\left(\frac{\partial \boldsymbol{b}_\nu}{\partial x^\mu} - \frac{\partial \boldsymbol{b}_\mu}{\partial x^\nu}\right) + \mathrm{i}g\left(\boldsymbol{a}_\nu M\boldsymbol{b}_\mu - \boldsymbol{a}_\mu M\boldsymbol{b}_\nu + \boldsymbol{f}_{\mu\nu}\boldsymbol{\phi}\right)\\
\overline{\boldsymbol{q}}_{\mu\nu}M^T &= \left(\frac{\partial \overline{\boldsymbol{b}}_\nu}{\partial x^\mu} - \frac{\partial \overline{\boldsymbol{b}}_\mu}{\partial x^\nu}\right)M^T - \mathrm{i}g\left(\overline{\boldsymbol{b}}_\mu M^T\boldsymbol{a}_\nu - \overline{\boldsymbol{b}}_\nu M^T\boldsymbol{a}_\mu + \overline{\boldsymbol{\phi}}\,\boldsymbol{f}_{\mu\nu}\right).
\end{aligned}
$$

The terms in parentheses in the first line of $\mathcal{L}_{3,\mathrm{KG}}$ can be regarded as the "minimum coupling rule" for the actual system. Under the inhomogeneous transformation prescription of the base fields from Eqs. (12) and the transformation rules of the gauge fields from Eqs. (19), the Lagrangian $\mathcal{L}_{3,\mathrm{KG}}$ is form-invariant. Moreover, the Lagrangian contains a term that is proportional to the square of the 4-vector gauge fields $\boldsymbol{b}_J$

$$
\overline{\boldsymbol{b}}^\alpha M^T M\,\boldsymbol{b}_\alpha,
$$

which represents a Proca mass term for an N-tuple of possibly charged bosons. Setting up the Euler-Lagrange equation for the gauge fields $\boldsymbol{b}_\mu$, we get

$$
\frac{\partial \boldsymbol{q}^{\mu\alpha}}{\partial x^\alpha} - \mathrm{i}gM^T\boldsymbol{a}_\alpha\left(M^T\right)^{-1}\boldsymbol{q}^{\mu\alpha} + M^T\left(\frac{\partial \boldsymbol{\phi}}{\partial x_\mu} - \mathrm{i}g\,\boldsymbol{a}^\mu\boldsymbol{\phi}\right) - M^TM\,\boldsymbol{b}^\mu = 0.
$$

We observe that this equation describes an N-tuple *massive* bosonic fields $b_{J\mu}$, in conjunction with their interactions with the *massless* gauge fields $a_{IJ\mu}$ and the base fields, ϕ_I.

For the case $N = 1$, hence for a single base field ϕ, the following twofold amended Klein-Gordon Lagrangian $\mathcal{L}_{3,\mathrm{KG}}$

$$
\mathcal{L}_{3,\mathrm{KG}} = \left(\frac{\partial \overline{\phi}}{\partial x_\alpha} + \mathrm{i}g\,\overline{\phi}\,a^\alpha - m\,\overline{b}^\alpha\right)\left(\frac{\partial \phi}{\partial x^\alpha} - \mathrm{i}g\,a_\alpha\phi - m\,b_\alpha\right) - \tfrac{1}{4}f^{\alpha\beta}\,f_{\alpha\beta} - \tfrac{1}{2}\overline{q}^{\alpha\beta}q_{\alpha\beta}
$$

is form-invariant under the combined local gauge transformation

$$\phi \mapsto \Phi \quad = \phi \, e^{\mathrm{i}\Lambda} + \varphi, \qquad a_\mu \mapsto A_\mu = a_\mu + \frac{1}{g}\frac{\partial \Lambda}{\partial x^\mu}$$
$$b_\mu \mapsto B_\mu = b_\mu \, e^{\mathrm{i}\Lambda} - \frac{\mathrm{i}g}{m}\left(a_\mu + \frac{1}{g}\frac{\partial \Lambda}{\partial x^\mu}\right)\varphi + \frac{1}{m}\frac{\partial \varphi}{\partial x^\mu}.$$

The field tensors then simplify to

$$f_{\mu\nu} = \frac{\partial a_\nu}{\partial x^\mu} - \frac{\partial a_\mu}{\partial x^\nu}$$
$$q_{\mu\nu} = \frac{\partial b_\nu}{\partial x^\mu} - \frac{\partial b_\mu}{\partial x^\nu} + \mathrm{i}g\left(a_\nu \, b_\mu - a_\mu \, b_\nu\right) + \frac{\mathrm{i}g}{m}\left(\frac{\partial a_\nu}{\partial x^\mu} - \frac{\partial a_\mu}{\partial x^\nu}\right)\phi$$
$$\overline{q}_{\mu\nu} = \frac{\partial \overline{b}_\nu}{\partial x^\mu} - \frac{\partial \overline{b}_\mu}{\partial x^\nu} - \mathrm{i}g\left(\overline{b}_\mu \, a_\nu - \overline{b}_\nu \, a_\mu\right) - \frac{\mathrm{i}g}{m}\overline{\phi}\left(\frac{\partial a_\nu}{\partial x^\mu} - \frac{\partial a_\mu}{\partial x^\nu}\right).$$

With $m^2 \, \overline{b}^\alpha b_\alpha$, this locally gauge-invariant Lagrangian contains a mass term for the complex bosonic 4-vector gauge field b_μ. The subsequent equation *massive* gauge field b_μ is thus

$$\frac{\partial q^{\mu\alpha}}{\partial x^\alpha} - \mathrm{i}g \, a_\alpha q^{\mu\alpha} + m\left(\frac{\partial \phi}{\partial x_\mu} - \mathrm{i}g \, a^\mu \phi\right) - m^2 b^\mu = 0.$$

From the transformation rule for the fields, the rule for the momenta $Q_{\mu\nu}$ follows as

$$Q_{\mu\nu} = q_{\mu\nu} \, e^{\mathrm{i}\Lambda(x)}.$$

It is then easy to verify that the field equation is indeed form-invariant under the above combined local transformation of the fields ϕ, a_μ, b_μ.

5. Conclusions

With the present paper, we have worked out a complete non-Abelian theory of *inhomogeneous* local gauge transformations. The theory was worked out as a canonical transformation in the realm of covariant Hamiltonian field theory. A particularly useful device was the definition of a gauge field *matrix* $\boldsymbol{a}_{IJ}$, with each matrix element representing a 4-vector gauge field. This way, the mutual interactions of base fields ϕ_I and both sets of gauge fields, $\boldsymbol{a}_{IJ}$ and $\boldsymbol{b}_J$, attain a straightforward algebraic representation as ordinary matrix products.

Not a single assumption or postulate needed to be incorporated in the course of the derivation. Moreover, no premise with respect to a particular "potential energy" function was required nor any draft on a "symmetry breaking" mechanism. The only restriction needed to render the theory consistent was to require the *mass matrix* to be *orthogonal*. Under *inhomogeneous* linear transformations of the base fields, the demand of local gauge invariance of the system's Hamiltonian actually requires the existence of *massive gauge fields*. Specifically, the formalism enforces to introduce both a set of massless gauge fields and a set of massive gauge fields.

The various mutual interactions of base and gauge fields that are described by the corresponding gauge-invariant Lagrangian $\mathcal{L}_3$ give rise to a variety of processes that can be used to test whether this beautiful formalism is actually reflected by nature.

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References

Cheng T-P and Li L-F 2000 *Gauge theory of elementary particle physics* (Oxford: Clarendon)
DeDonder Th 1930 *Théorie Invariantive Du Calcul des Variations* (Paris: Gaulthier-Villars & Cie)
Griffiths D 2008 *Introduction to Elementary Particles* (Weinheim: Wiley-VCH)
Higgs P W 1964 *Phys. Letters* **12** 132
José J V and Saletan E J 1998 *Classical Dynamics* (Cambridge: Cambridge University Press)
Kibble T W B 1967 *Phys. Rev.* **155** 1554
Ryder L 1996 *Quantum Field Theory, 2nd ed* (Cambridge: Cambridge University Press)
Struckmeier J 2009 Extended Hamilton-Lagrange formalism and its application to Feynman's path integral for relativistic quantum physics *Int. J. Mod. Phys. E* **18** 79, http://arxiv.org/abs/0811.0496 [quant-ph]
Struckmeier J and Reichau H 2012 General $\mathrm{U}(N)$ gauge transformations in the realm of covariant Hamiltonian field theory *Proc. Symp. on Exciting Physics: Quarks and Gluons/Atomic Nuclei/Biological Systems/Networks (Makutsi Safari Farm, South Africa) Interdisciplinary Lectures, Frankfurt Institute for Advanced Studies (FIAS)* (to be published by Heidelberg: Springer), http://arxiv.org/abs/1205.5754 [hep-th]
Struckmeier J 2013 Generalised $\mathrm{U}(N)$ gauge transformations in the realm of the extended covariant Hamilton formalism of field theory Extended Hamilton-Lagrange formalism *J. Phys. G: Nucl. Part. Phys.* **40** 015007, http://arxiv.org/abs/1206.4452 [nucl-th]
Weyl H 1919 Eine neue Erweiterung der Relativitätstheorie *Annalen der Physik IV Folge* **59** 101
Weyl H 1935 Geodesic Fields in the Calculus of Variation for Multiple Integrals *Annals of Mathematics* **36** 607